# Suggested Solution 6

(1) In the proof of Lusin's Theorem (Theorem 2.12), it was assumed that $f$ is non-negative, bounded and $A$ is compact. Complete the proof by showing the conclusion still holds when $f$ is finite a.e. and $A$ is of finite measure.

**Solution**: We divide the proof into three steps.

Step 1. Assume that $f$ is bounded and supported on a compact set $A$. Write $f = f^+ - f^-$. Then both $f^+$ and $f^-$ are bounded and supported on $A$. Then by what is proved in Theorem 2.12, the conclusion of Lusin's Theorem holds in this situation.

Step 2. Assume that $f$ is bounded and vanishes outside a measurable set $A$ with $\mu(A) < \infty$. Let $\epsilon > 0$ be fixed. By the regularity of $\mu$, there exists a compact set $K$ and an open set $G$ such that $K \subset A \subset G$ and $\mu(G \setminus K) < \frac{\epsilon}{2}$. By Urysohn's Lemma, there exists $h \in C_c(X)$ such that $K < h < G$.

Now we apply Step 1 to $f|_K$, we have there exists $g \in C_c(X)$ such that

$$\mu(\{x \in X : g(x) \neq f|_K(x)\}) < \frac{\epsilon}{2}.$$

Observe that $gh \in C_c(x)$, $gh \equiv g$ on $K$ and $gh \equiv 0$ outside $G$. Hence we have

$$\{x : g(x)h(x) \neq f(x)\} \subseteq \{x : g(x) \neq f|_K(x)\} \cup (G \setminus K).$$

Therefore,

$$\begin{aligned}\mu(\{x : g(x)h(x) \neq f(x)\}) &\leq \mu(\{x : g(x) \neq f|_K(x)\}) + \mu(G \setminus K) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Step 3. Assume that $f$ is finite a.e. and vanishes outside a measurable set $A$ with $\mu(A) < \infty$. For each $n \geq 1$, we define

$$f_n(x) := \begin{cases} f(x) & \text{if } |f(x)| \leq n, \\ n \cdot \text{sign} f(x) & \text{otherwise.} \end{cases}$$

Then we have $f_n(x) \to f(x)$ for every $x \in X$. Note that

$$\{x : f_n(x) \neq f(x)\} \subseteq \{x : |f(x)| > n\}.$$

Since $f$ is finite a.e., supported on $A$ and $\mu(A) < \infty$, we have

$$\mu(\{x : |f(x)| > n\}) \downarrow 0, \text{ as } n \to \infty.$$

Hence there exists $n_0$, such that

$$\mu(\{x : f_{n_0}(x) \neq f(x)\}) < \frac{\epsilon}{2}.$$

Apply the result of Step 2 to $f_{n_0}$, we get a $g \in C_c(X)$ such that

$$\mu(\{x : g(x) \neq f_{n_0}(x)\}) < \frac{\epsilon}{2}.$$

Note that

$$\{x : g(x) \neq f(x)\} \subseteq \{x : g(x) = f_{n_0}(x), f_{n_0}(x) \neq f(x)\} \cup \{x : g(x) \neq f_{n_0}(x)\}.$$

Hence we have $\mu(\{x : g(x) \neq f(x)\}) \leq \epsilon$, completing the proof.

(2) Let $\mu$ be a Riesz measure on $\mathbb{R}^n$. Show that for every measurable function $f$, there exists a sequence of continuous functions $\{f_n\}$ such that $f_n \to f$ almost everywhere.

**Solution**: For each $k \geq 1$, we define a set $B_k := \{x \in \mathbb{R}^n : |x| \leq k\}$ and a function

$$f_k(x) := \begin{cases} f(x) & \text{if } x \in B_k \text{ and } |f(x)| \leq k, \\ k \cdot \text{sign} f(x) & \text{if } x \in B_k \text{ and } |f(x)| > k, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to see that $f_k(x) \to f(x)$ at every $x \in \mathbb{R}^n$. Note that $f_k$ is bounded and supported on a set of finite measure, we can apply the result of Exercise (1) to get a $g_k \in C_c(\mathbb{R}^n)$, such that

$$\mu(\{x \in \mathbb{R}^n : f_k(x) \neq g_k(x)\}) < \frac{1}{2^k}.$$

Let $A_k = \{x \in \mathbb{R}^n : g_k(x) \neq f_k(x)\}$. Then by the Borel-Cantelli Lemma, we have for almost every $x \in \mathbb{R}^n$, $x \in A_k$ for finite many $k$. As a consequence, we have $g_k \to f$ a.e..

(3) Here we construct a Cantor-like set, or a Cantor set with positive measure, with positive measure by modifying the construction of the Cantor set as follows. Let $\{a_k\}$ be a sequence of positive numbers satisfying

$$\gamma \equiv \sum_{k=1}^{\infty} 2^{k-1} a_k < 1.$$

Construct the set $\mathcal{S}$ so that at the $k$th stage of the construction one removes $2^{k-1}$ centrally situated open intervals each of length $a_k$. Establish the facts:

(a) $\mathcal{L}^1(\mathcal{S}) = 1 - \gamma$,

(b) $\mathcal{S}$ is compact and nowhere dense.

(c) $\mathcal{S}$ is perfect hence uncountable.

Note. A set $A$ is perfect if for every $x \in A$ and $\epsilon > 0$, $(B_\epsilon(x) \setminus \{x\}) \cap A \neq \emptyset$, that is, every point in $A$ is an accumulation point of $A$. It is known that a perfect set must be uncountable.

**Solution**:

(a) As the intervals removed at the same stage or different stages are mutually disjoint, we have

$$\begin{aligned} \mathcal{L}^1(\mathcal{S}) &= 1 - \sum_{k=1}^{\infty} 2^{k-1} \text{ length of interval removed in the k th stage} \\ &= 1 - \sum_{k=1}^{\infty} 2^{k-1} a_k \\ &= 1 - \gamma. \end{aligned}$$

(b) Let $S_n$ be the set of points left in $[0,1]$ after the $n$-th level construction. Then $S_n$ is descending and $\mathcal{S} = \bigcap_{n=1}^{\infty} S_n$. Notice that $S_n$ is a union of $2^n$ mutually disjoint closed intervals hence is compact. Hence $\mathcal{S}$ is compact. The $2^n$ components of $S_n$ are of the same length

$$b_n = 2^{-n} \left( 1 - \sum_{k=1}^{n} 2^{k-1} a_k \right).$$

Clearly $b_n \to 0$ as $n \to \infty$. Hence $\mathcal{S}$ does not have an interior point, since otherwise $\mathcal{S}$ will contain an open interval which is also contained in every $S_n$, which is impossible since $b_n \to \infty$ as $n \to \infty$. Hence $\mathcal{S}$ is nowhere dense.

(c) If $x \in S$, then $x$ belongs some connected component of $S_n, \forall n \in \mathbb{N}$. Observe that the end points of the $2^n$ intervals of $S_n$ are in $S$, so $\exists y_n$ end point of one of the interval s.t.

$$|y_n - x| \le b_n \to 0 \text{ as } n \to \infty.$$

We have $S$ is a perfect set.

(4) Let $0 < \varepsilon < 1$. Construct an open set $G \subset [0,1]$ which is dense in $[0,1]$ but $\mathcal{L}^1(G) = \varepsilon$.

**Solution**: Similar to the construction of Cantor's familiar "middle thirds" set. Define $K_0 = [0,1]$ and inductively define $K_n \subset K_{n-1}$ by removing an open interval of length $2(1-\varepsilon)2^{-2n}$. By the construction each $K_n$ has $2^n$ connected components with length $a_n$ which satisfy

$$\begin{cases} a_n = \dfrac{1}{2}(a_{n-1} - 2\varepsilon 2^{-2n}), \quad n = 1, 2, \dots \\ a_0 = 1, \end{cases}$$

from which we get $a_n = (1-\varepsilon)2^{-n} + \varepsilon 2^{-2n}$. Thus

$$\mathcal{L}^1(K) = \lim_{n\to\infty} \mathcal{L}^1(K_n) = \lim_{n\to\infty} 2^n a_n = 1 - \varepsilon.$$

Take $G = [0,1] \setminus K$, then $\mathcal{L}^1(G) = \varepsilon$. On the other hand, $G$ is dense in $[0,1]$ since the interior of $K$ is empty.

(5) Let $A$ be the subset of $[0,1]$ which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Find $\mathcal{L}^1(A)$.

**Solution**: Let $B = \{0,1,2,3,5,6,7,8,9\}$, the set $F_0 = \{x \in [0,1] : x = 0.4a_2a_3\cdots, a_j = 0,1,2,\cdots,9\} = [\frac{4}{10}, \frac{5}{10}]$ is of Lebesgue measure $\frac{1}{10}$. Fix $y_1 \in B$, $|B| = 9^1 = 9$, the set $F_{y_1} = \{x \in [0,1] : x = 0.y_1 4 a_3 \cdots, a_j = 0,1,2,\cdots,9\, \forall j \ge 3\} = [\frac{y_1}{10} + \frac{4}{100}, \frac{y_1}{10} + \frac{5}{100}]$ is of Lebesgues measure $\frac{1}{100}$. Fix $(y_1, y_2) \in B^2$, $|B^2| = 9^2 = 81$, the set $F_{(y_1,y_2)} = \{x \in [0,1] : x = 0.y_1y_2 4 a_4 \cdots, a_j = 0,1,2,\cdots,9\, \forall j \ge 4\}$ is of measure $\frac{1}{1000}$. Continuing the process, we have

$$A = [0,1] \setminus (\bigcup_{n=1}^{\infty} \bigcup_{(y_1,y_2,\cdots,y_n)\in B^n} F_{(y_1,y_2,\cdots,y_n)} \cup F_0)$$

and as all $F_{(y_1,y_2,\cdots,y_n)}, F_0$ are disjoint, we have

$$\begin{aligned}\mathcal{L}^1(A) &= 1-\frac{1}{10}-\sum_{n=1}^{\infty}\sum_{(y_1,y_2,\cdots,y_n)\in B^n}\frac{1}{10^{n+1}}\\ &= 1-\frac{1}{10}-\sum_{n=1}^{\infty}\frac{9^n}{10^{n+1}}\\ &= 0.\end{aligned}$$

(6) Let $\mathcal{N}$ be a Vitali set in $[0,1]$. Show that $\mathcal{M}=[0,1]\setminus\mathcal{N}$ has measure 1 and hence deduce that

$$\mathcal{L}^1(\mathcal{N})+\mathcal{L}^1(\mathcal{M})>\mathcal{L}^1(\mathcal{N}\cup\mathcal{M}).$$

Remark: I have no idea what $\mathcal{L}^1(\mathcal{N})$ is, except that it is positive.

**Solution**: We first prove that every Lebesgue measurable subset of $\mathcal{N}$ must be of measure zero. Let $A$ be a Lebesgue measurable subset of $\mathcal{N}$, $\{A+q\}_{q\in\mathbb{Q}\cap[0,1)}$ is a sequence of disjoint measurable set contained inside $[-1,2]$. By translational invariance of Lebesgue measure,

$$\mathcal{L}^1(\bigcup_{q\in\mathbb{Q}\cap[0,1)}A+q)=\sum_{q\in\mathbb{Q}\cap[0,1)}\mathcal{L}^1(A+q)=\sum_{q\in\mathbb{Q}\cap[0,1)}\mathcal{L}^1(A)<\infty,$$

Therefore we must have

$$\mathcal{L}^1(A)=0.$$

We try to prove by contradiction, suppose there is an open set $G$ s.t. $\mathcal{L}^1(G)=1-\varepsilon<1$ and $G\supseteq\mathcal{N}^c$. Then $[0,1]\setminus G$ is a measurable subset of $\mathcal{N}$ satisfying

$$0<\varepsilon=\mathcal{L}^1([0,1])-\mathcal{L}^1(G)\le\mathcal{L}^1([0,1]\setminus G).$$

Contradicting to our previous result.

(7) Let $E$ be a subset of $\mathbb{R}$ with positive Lebsegue measure. Prove that for each $\alpha\in(0,1)$, there exists an open interval $I$ so that

$$\mathcal{L}^1(E\cap I)\ge\alpha\mathcal{L}^1(I).$$

It shows that $E$ contains almost a whole interval. Hint: Choose an open $G$ containing $E$

such that $\mathcal{L}^1(E) \geq \alpha\mathcal{L}^1(G)$ and note that $G$ can be decomposed into disjoint union of open intervals. One of these intervals satisfies our requirement.

**Solution**: As $\exists n \in \mathbb{N}$ s.t. $\mathcal{L}^1(E \cap (-n,n)) > 0$, WLOG we may assume that $E$ has finite outer measure, then $\forall \alpha \in (0,1)$, $\exists$ open $G$ s.t.$G \supseteq E$ and

$$\mathcal{L}^1(E) + \frac{(1-\alpha)}{\alpha}\mathcal{L}^1(E) \geq \mathcal{L}^1(G),$$

Hence

$$\mathcal{L}^1(E) \geq \alpha\mathcal{L}^1(G).$$

we can write $G = \bigcup_{i=1}^{\infty} I_i$ where $I_i$ are disjoint open intervals. Then one of these $I_i$ must satisfy the desired property, otherwise

$$\mathcal{L}^1(E) \leq \sum_{i=1}^{\infty} \mathcal{L}^1(E \cap I_i) < \alpha \sum_{i=1}^{\infty} \mathcal{L}^1(I_i) = \alpha\mathcal{L}^1(G) < \infty,$$

contradicting the above inequality.

(8) Let $E$ be a measurable set in $\mathbb{R}$ with respect to $\mathcal{L}^1$ and $\mathcal{L}^1(E) > 0$. Show that $E - E$ contains an interval $(-a,a)$, $a > 0$. Hint:

(a) $U$, $V$ open, with finite measure, $x \mapsto \mathcal{L}^1((x+U) \cap V)$ is continuous on $\mathbb{R}$.

(b) $A$, $B$ measurable, $\mu(A), \mu(B) < \infty$, then $x \mapsto \mathcal{L}^1((x+A) \cap B)$ is continuous. For $A \subset U$, $B \subset V$, try

$$\mathcal{L}^1((x+U) \cap V) - \mathcal{L}^1((x+A) \cap B)| \leq \mathcal{L}^1(U \setminus A) + \mathcal{L}^1(V \subset B).$$

(c) Finally, $x \mapsto \mathcal{L}^1((x+E) \cap E)$ is positive at 0 and if $(x+E) \cap E \neq \emptyset$, then $x \in E \setminus E$.

**Solution**:

(a) We prove the case when U is an open interval $I$, note for all subset $A, B$ of $\mathbb{R}$,

$$((x+A) \cap B) \setminus (((y+A) \cap B)) = (x+A) \setminus (y+A) \cap B.$$

Therefore

$$\left|\mathcal{L}^1((x+I) \cap V) - \mathcal{L}^1((y+I) \cap V)\right| \leq \mathcal{L}^1((x+I) \setminus (y+I)) + \mathcal{L}^1((y+I) \setminus (x+I)) \leq 4\left|x-y\right|.$$

the function is Lipschitz and continuous. In general $U$ can be written as countable union of disjoint open intervals $\{I_i\}$, as $\sum_{i=1}^{\infty} \ell(I_i) < \infty, \exists N$ s.t. for all $k \geq N$,

$$\sum_{i=k}^{\infty} \ell(I_i) < \varepsilon.$$

We have

$$\sum_{i=1}^{\infty} \mathcal{L}^1((x+I_i)\cap V) - \mathcal{L}^1((y+I_i)\cap V) \leq \sum_{i=1}^{k} \mathcal{L}^1((x+I_i)\cap V) - \mathcal{L}^1((y+I_i)\cap V) + 2\varepsilon < 3\varepsilon$$

for $x$ sufficiently close to $y$. Similarly

$$\sum_{i=1}^{\infty} \mathcal{L}^1((y+I_i)\cap V) - \mathcal{L}^1((x+I_i)\cap V) \leq 3\varepsilon.$$

We have the function $\mathcal{L}^1((x+U)\cap V)$ is continuous.

(b) Obviously , $((x+U)\cap V)\setminus((x+A)\cap B) \subseteq U\setminus A \cup V \setminus B$. Therefore, we have

$$0 \leq \mathcal{L}^1((x+U)\cap V) - \mathcal{L}^1((x+A)\cap B) \leq \mathcal{L}^1(U\setminus A) + \mathcal{L}^1(V\setminus B).$$

Note RHS is independent on $x, y$, so the result follow from outer regularity of Lebesgue measure.

(c) the function $\mathcal{L}^1((x+E)\cap E)$ is continuous and positive at 0, $\exists a > 0$ s.t the function remain positive on $(-a, a)$, i.e

$$(x+E)\cap E \neq \emptyset$$

and $\forall x \in (-a, a), \exists e_1 e_2 \in E$ s.t

$$x = e_1 - e_2 \in E - E.$$

Alternate proof. The following is a simple proof due to Karl Stromberg.

By the regularity of $\mathcal{L}^1$, for every $\varepsilon > 0$ there are a compact set $K \subset E$ and an open set $U \supset E$ such that

$$\mathcal{L}^1(K) + \varepsilon > \mathcal{L}^1(E) > \mathcal{L}^1(U) - \varepsilon.$$

For our purpose it is enough to choose $K$ and $U$ such that

$$2\mathcal{L}^1(K) > \mathcal{L}^1(U).$$

Since $K \subset U$, there is an open cover of $K$ that is contained in $U$. Since $K$ is compact, one can choose a small neighborhood $V$ of 0 such that

$$K + V \subset U.$$

Let $v \in V$, and suppose

$$(K + v) \cap K = \emptyset.$$

Then,

$$2\mathcal{L}^1(K) = \mathcal{L}^1(K + v) + \mathcal{L}^1(K) < \mathcal{L}^1(U),$$

contradicting our choice of $K$ and $U$. Hence for all $v \in V$ there exists $k_1, k_2 \in K \subset E$ such that

$$k_1 + v = k_2,$$

which means that $V \subset E - E$.

(9) Give an example of a continuous map $\phi$ and a measurable $f$ such that $f \circ \phi$ is not measurable. Hint: May use the function $h = x + g(x)$ where $g$ is the Cantor function as $\phi$.

**Solution**: Let $h = x + g(x)$ where $g$ is the Cantor function. Then $h : [0, 1] \to [0, 2]$ is a strictly monotonic and continuous map, so its inverse $\phi = h^{-1}$ is continuous too. Since $g$ is constant on every interval in the complement of C, one has that $h$ maps such an interval to an interval of the same length. Hence $\mu(h(C)) = 1$, where $C$ is the cantor set. Then $h(C)$ contains a non-measurable set $A$ due to Proposition 3.3. Let $B = \phi(A)$. Set $f = \chi_B$. Then $f \circ \phi$ is not measurable since its inverse image of 1 is $A$.