Exercise 9

Standard notations are in force. Many problems are taken from [R].

(1) Consider $L^p(\mathbb{R}^n)$ with the Lebesgue measure, $0 < p < \infty$. Show that $\|f+g\|_p \le \|f\|_p + \|g\|_p$ holds $\forall f, g$ implies that $p \ge 1$. Hint: For $0 < p < 1$, $x^p + y^p \ge (x+y)^p$.

(2) Consider $L^p(\mu)$, $0 < p < 1$. Then $\dfrac{1}{q} + \dfrac{1}{p} = 1$, $q < 0$.

(a) Prove that $\|fg\|_1 \ge \|f\|_p \|g\|_q$.

(b) $f_1, f_2 \ge 0$. $\|f+g\|_p \ge \|f\|_p + \|g\|_p$.

(c) $d(f,g) \overset{\text{def}}{=\!=} \|f-g\|_p^p$ defines a metric on $L^p(\mu)$.

(3) Let X be a metric space consisting of infinitely many elements and μ a Borel measure on X such that $\mu(B) > 0$ on any metric ball (i.e. $B = \{x : d(x, x_0) < \rho\}$ for some $x_0 \in X$ and $\rho > 0$. Show that $L^\infty(\mu)$ is non-separable.

Suggestion: Find disjoint balls $B_{r_j}(x_j)$ and consider $\chi_{B_{r_j}(x_j)}$.

(4) Show that $L^1(\mu)' = L^\infty(\mu)$ provided $(X, \mathfrak{M}, \mu)$ is σ-finite, i.e., $\exists X_j$, $\mu(X_j) < \infty$, such that $X = \bigcup X_j$.

Hint: First assume $\mu(X) < \infty$. Show that $\exists g \in L^q(\mu)$, $\forall q > 1$, such that

$$\Lambda f = \int fg \, d\mu, \quad \forall f \in L^p, \ p > 1.$$

Next show that $g \in L^\infty(\mu)$ by proving the set $\{x : |g(x)| \ge M + \varepsilon\}$ has measure zero $\forall \varepsilon > 0$. Here $M = \|\Lambda\|$.

(5) (a) For $1 \le p < \infty$, $\|f\|_p, \|g\|_p \le R$, prove that

$$\int \big||f|^p - |g|^p\big| \, d\mu \le 2pR^{p-1} \|f-g\|_p .$$

(b) Deduce that the map $f \mapsto |f|^p$ from $L^p(\mu)$ to $L^1(\mu)$ is continuous.

Hint: Try $|x^p - y^p| \le p|x-y|(x^{p-1} + y^{p-1})$.

(6) Optional. Let $\mathfrak{M}$ be the collection of all sets E in the unit interval $[0,1]$ such that either E or its complement is at most countable. Let μ be the counting measure on this σ-algebra

$\mathfrak{M}$. If $g(x) = x$ for $0 \le x \le 1$, show that g is not $\mathfrak{M}$-measurable, although the mapping

$$f \mapsto \sum x f(x) = \int f g \, d\mu$$

makes sense for every $f \in L^1(\mu)$ and defines a bounded linear functional on $L^1(\mu)$. Thus $(L^1)^* \neq L^\infty$ in this situation.

(7) Optional. Let $L^\infty = L^\infty(m)$, where m is Lebesgue measure on $I = [0, 1]$. Show that there is a bounded linear functional $\Lambda \neq 0$ on L^∞ that is 0 on $C(I)$, and therefore there is no $g \in L^1(m)$ that satisfies $\Lambda f = \int_I f g \, dm$ for every $f \in L^\infty$. Thus $(L^\infty)^* \neq L^1$.

(8) Prove Brezis-Lieb lemma for $0 < p \le 1$.

Hint: Use $|a + b|^p \le |a|^p + |b|^p$ in this range.

(9) Let $f_n, f \in L^p(\mu)$, $0 < p < \infty$, $f_n \to f$ a.e., $\|f_n\|_p \to \|f\|_p$. Show that $\|f_n - f\|_p \to 0$.

(10) Suppose μ is a positive measure on X, $\mu(X) < \infty$, $f_n \in L^1(\mu)$ for $n = 1, 2, 3, \ldots$, $f_n(x) \to f(x)$ a.e., and there exists $p > 1$ and $C < \infty$ such that $\int_X |f_n|^p \, d\mu < C$ for all n. Prove that

$$\lim_{n \to \infty} \int_X |f - f_n| \, d\mu = 0.$$

Hint: $\{f_n\}$ is uniformly integrable.

(11) We have the following version of Vitali's convergence theorem. Let $\{f_n\} \subset L^p(\mu)$, $1 \le p < \infty$. Then $f_n \to f$ in L^p-norm if and only if

(i) $\{f_n\}$ converges to f in measure,

(ii) $\{|f_n|^p\}$ is uniformly integrable, and

(iii) $\forall \varepsilon > 0$, $\exists$ measurable E, $\mu(E) < \infty$, such that $\int_{X \setminus E} |f_n|^p \, d\mu < \varepsilon$, $\forall n$.

I found this statement from PlanetMath. Prove or disprove it.

(12) Let $\{x_n\}$ be bounded in some normed space X. Suppose for Y dense in X', $\Lambda x_n \to \Lambda x$, $\forall \Lambda \in Y$ for some x. Deduce that $x_n \rightharpoonup x$.

(13) Consider $f_n(x) = n^{1/p} \chi(nx)$ in $L^p(\mathbb{R})$. Then $f_n \rightharpoonup 0$ for $p > 1$ but not for $p = 1$. Here $\chi = \chi_{[0,1]}$.

(14) Let $\{f_n\}$ be bounded in $L^p(\mu)$, $1 < p < \infty$. Prove that if $f_n \to f$ a.e., then $f_n \rightharpoonup f$. Is this result still true when $p = 1$?

(15) The construction of Cantor diagonal sequence. Let f_n be a sequence of real-valued functions defined on some set and $\{x_k\}$ a subset of this set. Suppose that there is some M such that $|f_n(x_k)| \leq M$ for all n, k. Show that there is a subsequence $\{f_{n_j}\}$ satisfying $\lim_{j\to\infty} f_{n_j}(x_k)$ exists for each x_k.