Exercise 6

Many problems are taken from [R].

(1) In the proof of Lusin's (Theorem 2.12) it was assumed that f is non-negative, bounded and A is compact. Complete the proof by showing the conclusion still holds when f is finite a.e. and A is of finite measure.

(2) Let μ be a Riesz measure on $\mathbb{R}^n$. Show that for every measurable function f, there exists a sequence of continuous function $\{f_n\}$ such that $f_n \to f$ almost everywhere.

(3) Here we construct a Cantor-like set, or a Cantor set with positive measure, by modifying the construction of the Cantor set as follows. Let $\{a_k\}$ be a sequence of positive numbers satisfying

$$\gamma \equiv \sum_{k=1}^{\infty} 2^{k-1} a_k < 1.$$

Construct the set $\mathcal{S}$ so that at the kth stage of the construction one removes 2^{k-1} centrally situated open intervals each of length a_k. Establish the facts:

(a) $\mathcal{L}^1(\mathcal{S}) = 1 - \gamma$,

(b) $\mathcal{S}$ is compact and nowhere dense,

(c) $\mathcal{S}$ is perfect and hence uncountable.

Note. A set A is perfect if for every $x \in A$ and $\varepsilon > 0$, $\big(B_\varepsilon(x) \setminus \{x\}\big) \cap A \neq \phi$, that is, every point in A is an accumulation point of A. It is known that a perfect set must be uncountable.

(4) Let $0 < \varepsilon < 1$. Construct an open set $G \subset [0,1]$ which is dense in $[0,1]$ but $\mathcal{L}^1(G) = \varepsilon$.

(5) Let A be the subset of $[0,1]$ which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Find $\mathcal{L}^1(A)$.

(6) Let $\mathcal{N}$ be a Vitali set in $[0,1]$. Show that $\mathcal{M} = [0,1] \setminus \mathcal{N}$ has measure 1 and hence deduce that

$$\mathcal{L}^1(\mathcal{N}) + \mathcal{L}^1(\mathcal{M}) > \mathcal{L}^1(\mathcal{N} \cup \mathcal{M}).$$

(7) Let E be a subset of $\mathbb{R}$ with positive Lebsegue measure. Prove that for each $\alpha \in (0,1)$, there exists an open interval I so that $\mathcal{L}^1(E \cap I) \geq \alpha \mathcal{L}^1(I)$. It shows that E contains almost

a whole interval. Hint: Choose an open G containing E such that $\mathcal{L}^1(E) \geq \alpha\mathcal{L}^1(G)$ and note that G can be decomposed into disjoint union of open intervals. One of these intervals satisfies our requirement.

(8) Let E be a measurable set in $\mathbb{R}$ with respect to $\mathcal{L}^1$ and $\mathcal{L}^1(E) > 0$. Show that $E-E$ contains an interval $(-a, a)$, $a > 0$. Hint:

(a) U, V open, with finite measure, $x \mapsto \mathcal{L}^1((x+U) \cap V)$ is continuous on $\mathbb{R}$.

(b) A, B measurable, $\mu(A), \mu(B) < \infty$, then $x \mapsto \mathcal{L}^1((x+A) \cap B)$ is continuous. For $A \subset U$, $B \subset V$, try

$$|\mathcal{L}^1((x+U) \cap V) - \mathcal{L}^1((x+A) \cap B)| \leq \mathcal{L}^1(U \setminus A) + \mathcal{L}^1(V \subset B).$$

(c) Finally, $x \mapsto \mathcal{L}^1((x+E) \cap E)$ is positive at 0 and if $(x+E) \cap E \neq \phi$, then $x \in E \setminus E$.

(9) Give an example of a continuous map ϕ and a measurable f such that $f \circ \phi$ is not measurable. Hint: The function $h = x + g(x)$ where g is the Cantor function is a continuous map from $[0, 1]$ to $[0, 2]$ with a continuous inverse.