Solution to MATH5011 homework 9

(1) Consider $L^p(\mathbb{R}^n)$ with the Lebesgue measure, $0 < p < \infty$. Show that $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ holds $\forall f, g$ implies that $p \geq 1$. Hint: For $0 < p < 1$, $x^p + y^p \geq (x+y)^p$.

Solution. Recall that in fact we have, for $x, y \geq 0$,

$$\begin{cases} x^p + y^p & \geq & (x+y)^p, \quad 0 < p < 1, \\ x^p + y^p & = & (x+y)^p, \quad p = 1, \\ x^p + y^p & \leq & (x+y)^p, \quad 1 < p < \infty. \end{cases}$$

Pick any $a, b \geq 0$ and define $f, g \in L^p(\mathbb{R}^n)$ by

$$f(x) = \begin{cases} a, & x \in [0,1]^n, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$g(x) = \begin{cases} b, & x \in [2,3]^n, \\ 0, & \text{otherwise.} \end{cases}$$

Simple calculations show that $\|f\|_p = a$, $\|g\|_p = b$ and $\|f+g\|_p = (a^p + b^p)^{1/p}$. Now the hypothesis implies $a^p + b^p \geq (a+b)^p$. Hence, $p \geq 1$.

(2) Consider $L^p(\mu)$, $0 < p < 1$. Then $\dfrac{1}{q} + \dfrac{1}{p} = 1$, $q < 0$.

(a) Prove that $\|fg\|_1 \geq \|f\|_p \|g\|_q$.

(b) $f_1, f_2 \geq 0$. $\|f+g\|_p \geq \|f\|_p + \|g\|_p$.

(c) $d(f,g) \overset{\text{def}}{=\!=} \|f-g\|_p^p$ defines a metric on $L^p(\mu)$.

Solution.

(a) Assume that $g > 0$ everywhere first. Applying Hölder's inequality with conjugate exponents $\widetilde{p} = \dfrac{1}{p}$ and $\widetilde{q} = \dfrac{1}{1-p} = \dfrac{\widetilde{p}}{\widetilde{p}-1}$, we have

$$\begin{aligned}
\left\||f|^p\right\|_1 &= \left\||fg|^{1/\widetilde{p}}|g|^{-1/\widetilde{p}}\right\|_1 \\
&\leq \left\||fg|^{1/p}\right\|_{\widetilde{p}} \left\||g|^{-1/p}\right\|_{\widetilde{q}} \\
&= \|fg\|_1^{1/\widetilde{p}} \left\||g|^{-1/(\widetilde{p}-1)}\right\|_1^{(\widetilde{p}-1)/\widetilde{p}} \\
&= \|fg\|_1^p \left\||g|^{-p/(1-p)}\right\|_1^{1-p}, \text{ so} \\
\left\||f|^p\right\|_1^{1/p} &\leq \|fg\|_1 \left\||g|^{-p/(1-p)}\right\|_1^{1/p-1} \\
&= \|fg\|_1 \left\||g|^q\right\|_1^{-1/q}, \text{ or} \\
\|f\|_p &\leq \|fg\|_1 \|g\|_q^{-1}, \text{ that is} \\
\|fg\|_1 &\geq \|f\|_p \|g\|_q .
\end{aligned}$$

For a general $g \geq 0$, apply the result to $g_\varepsilon = g + \varepsilon$ first and then let g_ε tend to g.

(b) Without loss of generality, we can assume $\|f+g\|_p \neq 0$. Using part (a), we have

$$\begin{aligned}
\|f+g\|_p^p &= \int (f+g)^p \, d\mu \\
&= \int f(f+g)^{p-1} \, d\mu + \int g(f+g)^{p-1} \, d\mu \\
&\geq \left(\|f\|_p + \|g\|_p\right) \left(\int (f+g)^{(p-1)\left(\frac{p}{p-1}\right)} \, d\mu\right)^{1-\frac{1}{p}} \\
&= \left(\|f\|_p + \|g\|_p\right) \|f+g\|_p^{p-1}, \text{ so} \\
\|f+g\|_p &\geq \|f\|_p + \|g\|_p .
\end{aligned}$$

(c) The fact that for $x, y \geq 0$ and $0 < p < 1$,

$$(x+y)^p \leq x^p + y^p$$

implies

$$\int |f+g|^p \, d\mu \leq \int |f|^p \, d\mu + \int |g|^p \, d\mu.$$

Hence, $d(f,g) \stackrel{\text{def}}{=} \|f-g\|_p^p$ defines a metric on $L^p(\mu)$.

(3) Let X be a metric space consisting of infinitely many elements and μ a Borel measure on X such that $\mu(B) > 0$ on any metric ball (i.e. $B = \{x : d(x, x_0) < \rho\}$ for some $x_0 \in X$ and $\rho > 0$. Show that $L^\infty(\mu)$ is non-separable.

Suggestion: Find disjoint balls $B_{r_j}(x_j)$ and consider $\chi_{B_{r_j}(x_j)}$.

Solution. Standard argument.

(4) Show that $L^1(\mu)' = L^\infty(\mu)$ provided $(X, \mathfrak{M}, \mu)$ is σ-finite, i.e., $\exists X_j$, $\mu(X_j) < \infty$, such that $X = \bigcup X_j$.

Hint: First assume $\mu(X) < \infty$. Show that $\exists g \in L^q(\mu)$, $\forall q > 1$, such that

$$\Lambda f = \int f g \, d\mu, \quad \forall f \in L^p,\ p > 1.$$

Next show that $g \in L^\infty(\mu)$ by proving the set $\{x : |g(x)| \geq M + \varepsilon\}$ has measure zero $\forall \varepsilon > 0$. Here $M = \|\Lambda\|$.

Solution.

Step 1. $\mu(X) < \infty$.

In this case, Hölder's inequality implies that a continuous linear functional Λ on $L^1(X)$ has a restriction to $L^p(X)$ which is again continuous since

$$|\Lambda f| \leq \|\Lambda\| \, \|f\|_1 \leq \|\Lambda\| \, \mu(X)^{1/q} \, \|f\|_p \tag{1}$$

for all $p \geq 1$. By the proof for $p > 1$ in the lecture notes, we have the existence of a unique $v_p \in L^q(X)$ such that $\Lambda f = \int v_p f \, d\mu$ for all $f \in L^p(X)$. Moreover, since $L^r(X) \subset L^p(X)$ for $r \geq p$ (by Hölder's inequality) the uniqueness of v_p implies that v_p is, in fact, independent of p, i.e. this function (which we now call v) is in every $L^r(X)$-space for $1 < r < \infty$.

If we now pick some conjugate exponents q and p with $p > 1$ and choose $f = |v|^{q-2}\overline{v}$ in (1), we obtain

$$\begin{aligned}\int |v|^q \, d\mu &= \Lambda f \\ &\leq \|\Lambda\| \, \mu(X)^{1/q} \left(\int |v|^{(q-1)p} \, d\mu \right)^{1/p} \\ &= \|\Lambda\| \, \mu(X)^{1/q} \, \|v\|_q^{q-1},\end{aligned}$$

and hence $\|v\|_q \leq \|\Lambda\| \, \mu(X)^{1/q}$ for all $q < \infty$. We claim that $v \in L^\infty(X)$; in fact $\|v\|_\infty \leq \|\Lambda\|$. Suppose that $\mu(\{x \in X : |v(x)| > \|\Lambda\| + \varepsilon\}) = M > 0$. Then $\|v\|_q \geq (\|\Lambda\| + \varepsilon) M^{1/q}$, which exceeds $\|\Lambda\| \, \mu(X)^{1/q}$ if q is big enough. Thus

$v \in L^\infty(X)$ and $\Lambda f = \int vf\,d\mu$ for all $f \in L^p(X)$ for any $p > 1$. If $f \in L^1(X)$ is given, then $\int |v||f|\,d\mu < \infty$. Replacing f by $f^k = f\chi_{\{x:|f(x)|\le k\}}$, we note that $|f^k| \le |f|$ and $f^k(x) \to f(x)$ pointwise as $k \to \infty$; hence, by dominated convergence, $f^k \to f$ in $L^1(X)$ and $vf^k \to vf$ in $L^1(X)$. Thus

$$\Lambda f = \lim_{k\to\infty} \Lambda f^k = \lim_{k\to\infty} \int vf^k\,d\mu = \int vf\,d\mu.$$

Step 2. $\mu(X) = \infty$.

The previous conclusion can be extended to the case that $\mu(X) = \infty$ but X is σ-finite. Then

$$X = \bigcup_{j=1}^{\infty} X_j$$

with $\mu(X_j)$ finite and with $X_j \cap X_k$ empty whenever $j \neq k$. Any $L^1(X)$ function f can be written as

$$f(x) = \sum_{j=1}^{\infty} f_j(x)$$

where $f_j = \chi_j f$ and χ_j is the characteristic function of X_j. $f_j \mapsto \Lambda f_j$ is then an element of $L^1(X_j)'$, and hence there is a function $v_j \in L^\infty(X_j)$ such that $\Lambda f_j = \int_{X_j} v_j f_j\,d\mu = \int_{X_j} v_j f\,d\mu$. The important point is that each v_j is bounded in $L^\infty(X_j)$ by the *same* $\|\Lambda\|$. Moreover, the function v, defined on all of X by $v(x) = v_j(x)$ for $x \in X_j$, is clearly measurable and bounded by $\|\Lambda\|$. Thus, we have $\Lambda f = \int_X vf\,d\mu$ by the countable additivity of the measure μ.

If there exist $v, w \in L^\infty(X)$ such that

$$\Lambda f = \int_X vf\,d\mu = \int_X wf\,d\mu, \quad \forall f \in L^1(X),$$

then

$$\int_X (v-w)f\,d\mu = 0, \quad \forall f \in L^1(X).$$

Suppose, on the contrary, that $(v - w) > 0$ on some $A \subset \mathfrak{M}$ with $0 < \mu(A) < \infty$. By taking $f = \chi_A$ one arrives at a contradiction. Thus, given $\Lambda \in L^1(X)$ there corresponds a unique $v \in L^\infty(X)$.

(5) (a) For $1 \le p < \infty$, $\|f\|_p, \|g\|_p \le R$, prove that

$$\int ||f|^p - |g|^p| \, d\mu \le 2pR^{p-1} \|f - g\|_p .$$

(b) Deduce that the map $f \mapsto |f|^p$ from $L^p(\mu)$ to $L^1(\mu)$ is continuous.

Hint: Try $|x^p - y^p| \le p|x-y|(x^{p-1} + y^{p-1})$.

Solution.

(a) Notice that $|x^p - y^p| \le p|x-y|(x^{p-1}+y^{p-1})$, which follows form the mean value theorem applying to $h(x) = x^p$. Then it follows easily from Hölder's inequality that

$$\int ||f|^p - |g|^p| \, d\mu \le 2pR^{p-1} \|f - g\|_p .$$

(b) This is a direct consequence of (a).

(6) Optional. Let $\mathfrak{M}$ be the collection of all sets E in the unit interval $[0,1]$ such that either E or its complement is at most countable. Let μ be the counting measure on this σ-algebra $\mathfrak{M}$. If $g(x) = x$ for $0 \le x \le 1$, show that g is not $\mathfrak{M}$-measurable, although the mapping

$$f \mapsto \sum x f(x) = \int fg \, d\mu$$

makes sense for every $f \in L^1(\mu)$ and defines a bounded linear functional on $L^1(\mu)$. Thus $(L^1)^* \ne L^\infty$ in this situation.

Solution. g is not $\mathfrak{M}$-measurable because $g^{-1}\left(\frac{1}{4}, \frac{3}{4}\right) = \left(\frac{1}{4}, \frac{3}{4}\right) \notin \mathfrak{M}$. The functional $\Lambda f = \sum x f(x)$ is clearly linear. To see that it is bounded, if $f \in L^1(\mu)$, then f is non-zero on an at most countable set $\{x_i\}$ and by integrability,

$$\sum_{i=1} |f(x_i)| < \infty.$$

Thus Λf is well defined as g is a bounded function. Hence the operator is bounded.

(7) Optional. Let $L^\infty = L^\infty(m)$, where m is Lebesgue measure on $I = [0,1]$. Show that there is a bounded linear functional $\Lambda \ne 0$ on L^∞ that is 0 on $C(I)$, and therefore there is no $g \in L^1(m)$ that satisfies $\Lambda f = \int_I fg \, dm$ for every $f \in L^\infty$. Thus $(L^\infty)^* \ne L^1$.

Solution. Method 1. For any $x \in I$ take $\Lambda_x f = g(x_+) - g(x_-)$ for all f such that $f = g$ a.e. for some function g such that the two one-sided limits $g(x_+)$ and $g(x_-)$ both exist. Then

$\|\Lambda_x - \Lambda_y\| \geq 1$ for $x \neq y$. With reference to the question, we can just take $x = 1/2$.

Method 2. Consider $\chi_{[0,\frac{1}{2}]} \in L^\infty \setminus C(I)$, as $C(I)$ is closed subspace in L^∞, by consequence of Hahn-Banach Theorem (thm 3.11 in p.38 of lecture notes on functional analysis.), there is non-zero bounded linear functional Λ on L^∞ which is zero on $C(I)$.

If there is $g \in L^1(m)$ that satisfies $\Lambda f = \int_I fg\, dm$ for every $f \in L^\infty$,

$$\Lambda f = \int_I fg\, dm = 0, \forall f \in C(I) \Rightarrow g = 0.$$

we have $\Lambda = 0$ which is impossible.

(8) Prove Brezis-Lieb lemma for $0 < p \leq 1$.

Hint: Use $|a + b|^p \leq |a|^p + |b|^p$ in this range.

Solution. Taking $g_n = f_n - f$ as a and f as b,

$$\big||f + g_n|^p - |g_n|^p\big| \leq |f|^p,$$

or,

$$-|f|^p \leq |f + g_n|^p - |g_n^p \leq |f|^p.$$

we have

$$-2|f|^p \leq |f + g_n|^p - |g_n|^p - |f|^p \leq 0$$

which implies

$$\big||f + g_n|^p - |g_n|^p - |f|^p\big| \leq 2|f|^p,$$

and result follows from Lebesgue dominated convergence theorem.

(9) Let $f_n, f \in L^p(\mu)$, $0 < p < \infty$, $f_n \to f$ a.e., $\|f_n\|_p \to \|f\|_p$. Show that $\|f_n - f\|_p \to 0$.

Solution. Using the Brezis-Lieb lemma for $0 < p < \infty$, we have

$$\begin{aligned}
\|f_n - f\|_p^p &= \int_X |f_n - f|^p\, d\mu \\
&\leq \int_X (|f_n - f|^p - (|f_n|^p - |f|^p))\, d\mu + \int_X (|f_n|^p - |f|^p)\, d\mu \\
&\leq \int_X ||f_n - f|^p - (|f_n|^p - |f|^p)\,|\, d\mu + \left(\|f_n\|_p^p - \|f\|_p^p\right) \\
&\to 0
\end{aligned}$$

as $n \to \infty$.

(10) Suppose μ is a positive measure on X, $\mu(X) < \infty$, $f_n \in L^1(\mu)$ for $n = 1, 2, 3, \dots$, $f_n(x) \to f(x)$ a.e., and there exists $p > 1$ and $C < \infty$ such that $\int_X |f_n|^p \, d\mu < C$ for all n. Prove that

$$\lim_{n\to\infty} \int_X |f - f_n| \, d\mu = 0.$$

Hint: $\{f_n\}$ is uniformly integrable.

Solution. By Vitali's convergence Theorem, it suffices to prove that $\{f_n\}$ is uniformly integrable. Let q be conjugate to p. By Hölder inequality,

$$\begin{aligned} \int_E |f_n| d\mu &\leq \|f_n\|_p \{\mu(E)\}^{\frac{1}{q}} \\ &\leq C^{\frac{1}{p}} \{\mu(E)\}^{\frac{1}{q}}, \end{aligned}$$

for any measurable E. Now the result follows easily.

(11) We have the following version of Vitali's convergence theorem. Let $\{f_n\} \subset L^p(\mu)$, $1 \leq p < \infty$. Then $f_n \to f$ in L^p-norm if and only if

(i) $\{f_n\}$ converges to f in measure,

(ii) $\{|f_n|^p\}$ is uniformly integrable, and

(iii) $\forall \varepsilon > 0$, there exists a measurable E, $\mu(E) < \infty$, such that $\int_{X\setminus E} |f_n|^p \, d\mu < \varepsilon$, $\forall n$.

I found this statement from PlanetMath. Prove or disprove it.

Solution. Let $\varepsilon > 0$. By (iii), there exists a set E of finite measure (WLOG assume positive measure) such that

$$\int_{\widetilde{E}} |f_n|^p < \varepsilon.$$

Since $\{f_n\}$ converges to f in measure, there is a subsequence $\{f_{n_k}\}$ which converges to f pointwisely a.e.. By Fatou's Lemma,

$$\int_{\widetilde{E}} |f|^p < \varepsilon.$$

By (ii), there exists $\delta > 0$ such that whenever $\mu(A) < \delta$,

$$\int_A |f_n|^p < \varepsilon^{\frac{1}{p}};$$

WLOG, by choosing a smaller δ, we may assume further whenever $\mu(A) < \delta$

$$\int_A |f|^p < \varepsilon^{\frac{1}{p}}$$

because there is a subsequence $\{f_{n_k}\}$ which converges to f pointwisely a.e. and we can apply Fatou's Lemma, By (i), there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$\mu\{x \in E : \Big|(f_n - f)(x)\Big|^p \geq \frac{\varepsilon}{\mu(E)}|\} < \delta.$$

Now, for $n \geq \mathbb{N}$, define $A_n = \{x \in E : |(f_n - f)(x)|^p \geq \frac{\varepsilon}{\mu(E)}\}$ and $B_n = E \setminus A_n$, and we have

$$\begin{aligned}
\int |f_n - f|^p &= \int_{\widetilde{E}} |f_n - f|^p + \int_E |f_n - f|^p \\
&< 2^p\varepsilon + \int_{A_n} |f_n - f|^p + \int_{B_n} |f_n - f|^p \\
&< 2^p\varepsilon + \left(\int_{A_n} |f_n|^p + \int_{A_n} |f|^p\right)^p + \varepsilon \\
&< 2^p\varepsilon + 2^p\varepsilon + \varepsilon = (2^{p+1} + 1)\varepsilon.
\end{aligned}$$

This completes the proof.

(12) Let $\{x_n\}$ be bounded in some normed space X. Suppose for Y dense in X', $\Lambda x_n \to \Lambda x$, $\forall \Lambda \in Y$ for some x. Deduce that $x_n \rightharpoonup x$.

Solution. Since $\{x_n\}$ is bounded, there exists $M > 0$ such that $\|x_n\| \leq M$. Write $M_1 = \max\{M, \|x\|\}$.

Given $\varepsilon > 0$ and $\Lambda \in X'$, choose $\Lambda_1 \in Y$ such that $\|\Lambda - \Lambda_1\| < \dfrac{\varepsilon}{3M_1}$ and choose N large such that $|\Lambda x_n - \Lambda x| < \dfrac{\varepsilon}{3}$. Then

$$\begin{aligned}
|\Lambda x_n - \Lambda x| &= |\Lambda x_n - \Lambda_1 x_n| + |\Lambda_1 x_n - \Lambda_1 x| + |\Lambda_1 x - \Lambda x| \\
&\leq \frac{\varepsilon}{3M_1} M + \frac{\varepsilon}{3} + \frac{\varepsilon}{3M_1} \|x\| \\
&< \varepsilon.
\end{aligned}$$

(13) Consider $f_n(x) = n^{1/p}\chi(nx)$ in $L^p(\mathbb{R})$. Then $f_n \rightharpoonup 0$ for $p > 1$ but not for $p = 1$. Here $\chi = \chi_{[0,1]}$.

Solution. For $1 < p < \infty$, let q be the conjugate exponent and let $g \in L^q(\mathbb{R})$. By Hölder's inequality and Lebesgue's dominated convergence theorem,

$$\begin{aligned}\int_{\mathbb{R}} f_n g\,dx &= \int_0^{\frac{1}{n}} n^{1/p} g(x)\,dx \\ &\le \left(\int_0^{\frac{1}{n}} (n^{1/p})^p\,dx\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{n}} |g(x)|^q\,dx\right)^{\frac{1}{q}} \\ &\le \left(\int_{\mathbb{R}} \chi_{[0,\frac{1}{n}]} |g(x)|^q\,dx\right)^{\frac{1}{q}} \\ &\to 0\end{aligned}$$

as $n \to \infty$. Hence, $f_n \rightharpoonup 0$.

For $p = 1$, take $g \equiv 1$ in $L^\infty(\mathbb{R})$. Then

$$\int_{\mathbb{R}} f_n g\,dx = n \int_0^{\frac{1}{n}} dx = 1.$$

Hence, $f_n \not\rightharpoonup 0$.

(14) Let $\{f_n\}$ be bounded in $L^p(\mu)$, $1 < p < \infty$. Prove that if $f_n \to f$ a.e., then $f_n \rightharpoonup f$. Is this result still true when $p = 1$?

Solution. It suffices to show that for any $g \in L^q(\mu)$,

$$\int (f_n - f) g d\mu \to 0 \text{ as } n \to \infty.$$

By Prop 4.14 the density theorem, we may consider the case where g is a simple function with finite support. Let E be a finite measure set such that $g = 0$ outside E and $M > 0$ be bound of g. By a previous problem, $\{f_n, f\}$ is uniformly integrable, for all $\varepsilon > 0, \exists \delta > 0$, s.t. for any A measurable s.t $\mu(A) < \delta$,

$$\int_A |h| d\mu < \varepsilon, h = f_n \text{ or } f.$$

By Egorov's Theorem, there is a measurable B s.t $\mu(E \setminus B) < \delta$ and f_n converges uniformly to f on B. Hence

$$\begin{aligned}\left|\int (f_n - f) g d\mu\right| &= \left|\int_E (f_n - f) g d\mu\right| \\ &= \left|\int_{E\setminus B} (f_n - f) g d\mu\right| + \left|\int_B (f_n - f) g d\mu\right| \\ &< 2M\varepsilon + \left|\int_B (f_n - f) g d\mu\right| \\ &< (2M+1)\varepsilon, \text{ for large n }.\end{aligned}$$

An alternate approach is, using the L^p-boundedness, a subsequence of f_n weakly converges to some $g \in L^p(\mu)$. Then a convex combination of this subsequence converges strongly to g. Hence it has a subsequence converges pointwisely to g. On the other hand, the whole sequence converges pointwisely to f. So $g = f$. We have shown that every weakly convergent subsequence of $\{f_n\}$ must converge pointwisely to f. Now, suppose that f_n does not converge weakly to f. There are $\rho > 0$ and $g \in L^q$, such that

$$\left|\int f_{n_k} g d\mu - \int f g d\mu\right| > \rho \ , \quad \forall n_k$$

for some subsequence f_{n_k}. But we can find a subsequence from this subsequence which converges weakly to f, contradiction holds.

For p=1, the result is false by the last problem.

(15) The construction of Cantor diagonal sequence. Let f_n be a sequence of real-valued functions defined on some set and $\{x_k\}$ a subset of this set. Suppose that there is some M such that $|f_n(x_k)| \leq M$ for all n, k. Show that there is a subsequence $\{f_{n_j}\}$ satisfying $\lim_{j\to\infty} f_{n_j}(x_k)$ exists for each x_k.

Solution. Let $A = \{x_j\}, j \geq 1$. Since $\{f_n(x_1)\}$ is a bounded sequence, we can extract a subsequence $\{f_n^1\}$ such that $\{f_n^1(x_1)\}$ is convergent. Next, as $\{f_n^1(x_2)\}$ is bounded, it has a subsequence $\{f_n^2\}$ such that $\{f_n^2(x_2)\}$ is convergent. Keep doing in this way, we obtain sequences $\{f_n^j\}$ satisfying (i) $\{f_n^{j+1}\}$ is a subsequence of $\{f_n^j\}$ and (ii) $\{f_n^j(x_1)\}, \{f_n^j(x_2)\}, \cdots, \{f_n^j(x_j)\}$ are convergent. Then the diagonal sequence $\{g_n\}$, $g_n = f_n^n$, for all $n \geq 1$, is a subsequence of $\{f_n\}$ which converges at every x_j.