## Solution to MATH5011 homework 8

(1) Let $f, g \in L^{p}(\mu), 1<p<\infty$. Show that the function

$$
\Phi(t)=\int_{X}|f+t g|^{p} d \mu
$$

is differentiable at $t=0$ and

$$
\Phi^{\prime}(0)=p \int_{X}|f|^{p-2} f g d \mu .
$$

Hint: Use the convexity of $t \mapsto|f+t g|^{p}$ to get

$$
|f+t g|^{p}-|f|^{p} \leq t\left(|f+g|^{p}-|f|^{p}\right), \quad t>0
$$

and a similar estimate for $t<0$.
Solution. Recall that for any convex function $\varphi$ defined on $[0,1]$, one has the elementary inequality

$$
\frac{\varphi(t)-\varphi(0)}{t-0} \leq \frac{\varphi(1)-\varphi(0)}{1-0}, \quad \forall t \in(0,1),
$$

which could be deduced from the definition of convexity. For $p>1, x \in X$, the function $\varphi(t)=|f(x)+\operatorname{tg}(x)|^{p}$ is differentiable and convex whenever $f(x)$ and $g(x)$ are finite, which can be seen from $\varphi^{\prime \prime}(t) \geq 0$. Applying the inequality above to this particular convex function, We have

$$
\frac{1}{t}\left\{|f+t g|^{p}-|f|^{p}\right\} \leq|f+g|^{p}-|f|^{p}, \forall t \in(0,1)
$$

By replacing $t$ with $-t$, we obtain a similar inequality

$$
|f|^{p}-|f-g|^{p} \leq \frac{1}{t}\left\{|f+t g|^{p}-|f|^{p}\right\}, \forall t \in(-1,0) .
$$

Now the desired result follows from an application of Lebesgue's dominated convergence theorem.
(2) Suppose $f$ is a measurable function on $X, \mu$ is a positive measure on $X$, and

$$
\varphi(p)=\int_{X}|f|^{p} d \mu=\|f\|_{p}^{p} \quad(0<p<\infty) .
$$

Let $E=\{p: \varphi(p)<\infty\}$. Assume $\|f\|_{\infty}>0$.
(a) If $r<p<s, r \in E$, and $s \in E$, prove that $p \in E$.
(b) Prove that $\log \varphi$ is convex in the interior of $E$ and that $\varphi$ is continuous on $E$.
(c) By (a), $E$ is connected. Is $E$ necessarily open? Closed? Can $E$ consist of a single point? Can $E$ be any connected subset of $(0, \infty)$ ?
(d) If $r<p<s$, prove that $\|f\|_{p} \leq \max \left(\|f\|_{r},\|f\|_{s}\right)$. Show that this implies the inclusion $L^{r}(\mu) \cap L^{s}(\mu) \subset$ $L^{p}(\mu)$.
(e) Assume that $\|f\|_{r}<\infty$ for some $r<\infty$ and prove that

$$
\|f\|_{p} \rightarrow\|f\|_{\infty} \quad \text { as } p \rightarrow \infty
$$

## Solution.

(a) Write $p=\lambda r+(1-\lambda) s$ for $0<\lambda<1$. By Hölder's inequality,

$$
\int_{X}|f|^{p} d \mu=\int_{X}|f|^{\lambda r}|f|(1-\lambda) s d \mu \leq\left(\int_{X}|f|^{r} d \mu\right)^{\lambda}\left(\int_{X}|f|^{s} d \mu\right)^{1-\lambda}
$$

which shows that $\varphi$ is finite on $[r, z]$. It follows that $E$ is an interval.
(b) Rewrite the inequality above as

$$
\varphi(\lambda r+(1-\lambda) s) \leq \varphi(r)^{\lambda} \cdot \varphi(s)^{1-\lambda}, \quad(0<\lambda<1)
$$

It is also true for $\lambda=0,1$. Hence for all $\lambda \in[0,1]$,

$$
\log \varphi(\lambda r+(1-\lambda) s) \leq \lambda \log \varphi(r)+(1-\lambda) \log \varphi(s)
$$

since $\log$ is increasing. Thus $\log \varphi(p)$ is convex on $[r, s]$. Hence $\varphi(x)$ is continuous in the interior of $E$. It follows form monotonicity applying to $\chi_{|f|>1} f$ and $\chi_{|f| \leq 1} f$ that $\varphi(x)$ is also continuous on $\partial E$.
(c) Let $X=(0, \infty)$ with the Lebesgue measure. $E$ can be any connected subset of $(0, \infty)$. The basic functions to consider are of the form $x^{k}$ and $x^{k}|\log x|^{m}$ near $x=0$ and $x=\infty$. Define

$$
\begin{array}{r}
g_{k}(x)=x^{k} \chi_{(0,1 / 2]}(x), \\
h_{k}(x)=x^{k} \chi_{(2, \infty)}(x), \\
g_{k, m}(x)=x^{k}|\log x|^{m} \chi_{(0,1 / 2]}(x), \\
h_{k, m}(x)=x^{k}|\log x|^{m} \chi_{(2, \infty)}(x),
\end{array}
$$

It is easy to see that $\int_{X} g_{k} d x<\infty$ iff $k>-1$ and $\int_{X} h_{k} d x<\infty$ iff $k<-1$. Since $|\log x| \leq C_{\varepsilon} e^{-\varepsilon}$ for $0 \leq x \leq 1$ and all $\epsilon>0, \int_{X} g_{k, m} d x$ is finite for $k>-1$ and infinite for $k>-1$. For $k=-1$, direct computations by substituting $u=\log x$ yield

$$
\int_{X} g_{k, m} d x=\int_{0}^{1 / 2} x^{-1}|\log x|^{m} d x=\int_{\log 2}^{\infty} u^{m} d u
$$

which is finite iff $m<-1$. Similarly, one can show $\int_{X} h_{k, m} d x$ is finite for $k>-1$ and infinite for $k>-1$.

If $k=-1$, the integral is finite if and only if $m<-1$. Note that $g_{k}^{p}=g_{p k}, g_{k, m}^{p}=g_{p k, p m}$ and similarly for $h$. Now for $f=g_{-1,-2}+h_{-1,-2}$, one has $E=1$. For $E=\emptyset$, take $f=g_{-1}+h_{-1}$. To get $E=(0, \infty)$, one may take $f=e^{-|x|}$. For $E=[1, p)$, take $f=g_{-1 / p}+h_{-1,-2}$. Similarly it is easy to see that $E$ can be any connected subset of $(0, \infty)$ for choosing $f$ properly.
(d) The inequality in (a) implies $\|f\|_{p} \leq \max \left(\|f\|_{r},\|f\|_{s}\right.$ ). Obviously, if $\|f\|_{r}<\infty$ and $\|f\|_{s}<\infty$, then $\|f\|_{p}<\infty$. Thus $L^{r}(\mu) \cap L^{s}(\mu) \subset L^{p}(\mu)$.
(e) Denote $E_{a}:=\{x: a \leq|f(x)|\}$ for every $0<a<\|f\|_{\infty}$, then $0<\mu\left(E_{a}\right)<\infty$. $\quad\left(\|f\|_{r}<\infty\right.$ implies $\mu\left(E_{a}\right)<\infty$.) Thus

$$
\|f\|_{p}=\left(\int_{X}|f|^{p} d \mu\right)^{1 / p} \geq\left(\int_{E_{a}}|f|^{p} d \mu\right)^{1 / p} \geq a\left(\mu\left(E_{a}\right)\right)^{1 / p}
$$


On the other hand, for $p>r$,

$$
\|f\|_{p}=\left(\int_{X}|f|^{p-r}|f| r d \mu\right)^{1 / p} \leq\|f\|_{r}^{r / p}\|f\|_{\infty}^{1-r / p}
$$

which implies $\varlimsup_{p \rightarrow \infty}\|f\|_{p} \leq\|f\|_{\infty}$. In conclusion, we have

$$
\lim _{p \rightarrow \infty}\|f\|_{\infty}=\|f\|_{\infty}
$$

(3) Assume, in addition to the hypothesis of the last problem, that

$$
\mu(X)=1
$$

(a) Prove that $\|f\|_{r} \leq\|f\|_{s}$ if $0<r<s \leq \infty$.
(b) Under what conditions does it happen that $0<r<s \leq \infty$ and $\|f\|_{r}=\|f\|_{s}<\infty$ ?
(c) Prove that $L^{r}(\mu) \supset L^{s}(\mu)$ if $0<r<s$. Under what conditions do these two spaces contain the same functions?
(d) Assume that $\|f\|_{r}<\infty$ for some $r>0$, and prove that

$$
\lim _{p \rightarrow 0}\|f\|_{p}=\exp \left\{\int_{X} \log |f| d \mu\right\}
$$

if $\exp \{-\infty\}$ is defined to be 0 .

## Solution.

(a) If $s<\infty$, the conclusion from from Hölder's inequality,

$$
\int_{X}|f|^{r} d \mu \leq\left(\int_{X}|f|^{s} d \mu\right)^{r / s}\left(\int_{X} 1 d \mu\right)^{1-r / s}=\|f\|_{s}^{r}
$$

If $s=\infty$, the desired result follows from

$$
\|f\|_{r} \leq\|f\|_{\infty}\left(\int_{X} 1 d \mu\right)^{1 / r}=\|f\|_{\infty}
$$

(b) From the equality sign characterization in the Hölder inequality it is easy to see that $\|f\|_{r}=\|f\|_{s}<\infty$ if and only if $|f|=\|f\|_{\infty}<\infty$ a.e..
(c) We claim that under the condition $\mu(X)<\infty, L^{r}(\mu)=L^{s}(\mu)$ for $0 \leq r<s \leq \infty$ if and only if the following property (call it $L$ ) holds:

There exists $\varepsilon_{0}>0$ such that for any measurable set $E \subset X$ with $\mu(E)>0$ we have $\mu(E)>\varepsilon_{0}$.
In fact, if Property $L$ holds, let $f \in L^{r}(\mu)$ and denote $E_{n}:=\{x:|f| \geq n\}$. Then there exists $n_{0} \in \mathbb{N}$ such that $\mu\left(E_{n_{0}}\right)=0$ and thus $f \in L^{\infty}(\mu)$. Otherwise for all $n, \mu\left(E_{n}\right)>0$. Thus $\mu(\{x:|f(x)|=\infty\}) \geq$ $\lim _{n \rightarrow \infty} \mu\left(E_{n}\right) \geq \varepsilon_{0}$ and then $\|f\|_{r}=\infty$, a contradiction.
Conversely, suppose there is a sequence of measurable sets $\left\{E_{n}\right\}$ with $0<\mu\left(E_{n}\right)<3^{-n}$. Without loss of generality, $E_{n}$ are mutually disjoint. Denote $a_{n}:=\mu\left(E_{n}\right)$ and define

$$
f= \begin{cases}\sum_{n=1}^{\infty} a_{n}^{-1 / s} \chi_{E_{n}}, & \text { if } s<\infty \\ \sum_{n=1}^{\infty} a_{n}^{-\frac{1}{2 r}} \chi_{E_{n}}, & \text { if } s=\infty\end{cases}
$$

Then $f \in L^{r}$ but $f \notin L^{s}$. The proof is completed.
(d) Note $x-1-\log x \geq 0$ on $[0, \infty)$ implies that

$$
\int_{\{|f|>1\}} \log |f| d \mu<\infty .
$$

If $\mu(\{|f|=0\})>0$, it suffices to proves the equality by showing $\lim _{p \rightarrow 0}\|f\|_{p}=0$. There is a small $s>1$, with $s^{\prime}$ be its conjugate s.t.

$$
\begin{aligned}
\|f\|_{p} & =\left\{\int_{X}|f|^{p} \chi_{\{|f|>0\}} d \mu\right\}^{\frac{1}{p}} \\
& \leq(\mu\{|f|>0\})^{\frac{1}{s^{\prime}}}\|f\|_{s p} \text { by Hölder inequality } \\
& \leq(\mu\{|f|>0\})^{\frac{1}{s^{\frac{1}{p}}}\|f\|_{r} \rightarrow 0 \text { as } p \rightarrow 0}
\end{aligned}
$$

We may suppose $\infty>|f|>0$ a.e. By Jensen's inequality, we have

$$
\log \|f\|_{p}=\frac{1}{p} \log \int_{X}|f|^{p} d \mu \geq \frac{1}{p} \int_{X} \log |f|^{p} d \mu=\int_{X} \log |f| d \mu .
$$

On the other hand, $x-1-\log x \geq 0$ on $[0, \infty)$ implies $\frac{\|f\|_{p}^{p}-1}{p} \geq \log \|f\|_{p}$. Thus

$$
\int_{X} \log |f| d \mu \leq \log \|f\|_{p} \leq \int_{X} \frac{|f|^{p}-1}{p} d \mu
$$

since $\mu(X)=1$. Note that by convexity of the map $p \mapsto|f|^{p}$ we have $\frac{|f|^{p}-1}{p}$ is increasing in $p$, which implies $\frac{|f|^{p}-1}{p} \leq \frac{|f|^{r}-1}{r} \in L^{1}(\mu)$ and $\lim _{p \rightarrow 0} \frac{|f|^{p}-1}{p}=\log |f|$. By Lebesgue's dominated convergence theorem for $|f|>1$ and monotone convergence theorem for $|f|<1$, we have

$$
\lim _{p \rightarrow 0} \int_{X} \frac{|f|^{p}-1}{p} d \mu=\lim _{p \rightarrow 0} \int_{\{|f| \geq 1\}} \frac{|f|^{p}-1}{p} d \mu+\lim _{p \rightarrow 0} \int_{\{|f|<1\}} \frac{|f|^{p}-1}{p} d \mu=\int_{X} \log |f| d \mu .
$$

Thus by sandwich rule

$$
\lim _{p \rightarrow 0}\|f\|_{p}=\exp \left\{\int_{X} \log |f| d \mu\right\}
$$

(4) For some measures, the relation $r<s$ implies $L^{r}(\mu) \subset L^{s}(\mu)$; for others, the inclusion is reversed; and there are some for which $L^{r}(\mu)$ does not contain $L^{s}(\mu)$ is $r \neq s$. Give examples of these situations, and find conditions on $\mu$ under which these situations will occur.

## Solution.

First, we give examples of these situations:
(a) For $X=[0,1]$ with usual Lebesgue measure, we have $L^{r}(\mu) \supset L^{s}(\mu)$ if $r<s$.
(b) For $X=\mathbb{N}$ with counting measure, we have $L^{r}(\mu) \subset L^{s}(\mu)$ if $r<s$.
(c) For $X=\mathbb{R}$ with usual Lebesgue measure, we have $L^{r}(\mu) \neq L^{s}(\mu)$ if $r \neq s$.

Second, we give simple conditions on $\mu$ under which these situations occur correspondingly:
(a) $\mu(X)<\infty$.
(b) Property $L$ in 6(c) holds.
(c) $\mu(X)=\infty$ and Property $L$ in 6(c) fails to hold.
(5) Suppose $\mu(\Omega)=1$, and suppose $f$ and $g$ are positive measurable functions on $\Omega$ such that $f g \geq 1$. Prove that

$$
\int_{\Omega} f d \mu \cdot \int_{\Omega} g d \mu \geq 1
$$

Solution. Since $f g \geq 1$, we have $\sqrt{f g} \geq 1$ and so by Hölder's inequality,

$$
1 \leq \int_{\Omega} \sqrt{f} \sqrt{g} d \mu \leq\left(\int_{\Omega} f d \mu\right)^{1 / 2}\left(\int_{\Omega} g d \mu\right)^{1 / 2}=\left(\int_{\Omega} f d \mu \cdot \int_{\Omega} g d \mu\right)^{1 / 2}
$$

(6) Suppose $\mu(\Omega)=1$ and $h: \Omega \rightarrow[0, \infty]$ is measurable. If

$$
A=\int_{\Omega} h d \mu
$$

prove that

$$
\sqrt{1+A^{2}} \leq \int_{\Omega} \sqrt{1+h^{2}} d \mu \leq 1+A .
$$

If $\mu$ is Lebesgue measure on $[0,1]$ and if $h$ is continuous, $h=f^{\prime}$, the above inequalities have a simple geometric interpretation. From this, conjecture (for general $\Omega$ ) under what conditions on $h$ equality can hold in either of the above inequalities, and prove your conjecture.

Solution. The function $\phi(x)=\sqrt{1+x^{2}}$ is convex since its second derivative is always positive. Hence the first inequality follows from Jensen's inequality. The second equality follows from $|\Omega|=1$ and $\sqrt{1+x^{2}} \leq 1+x$ for all $x \geq 0$.

In the case that $\Omega=[0,1]$ with $\mu$ the Lebesgue measure and $h=f^{\prime}$ is continuous, then $\int_{0}^{1} \sqrt{1+\left(f^{\prime}\right)^{2}} d x$ is the arc length of the graph of $f$. Then $A=f(1)-f(0)$. The first inequality means that the straight line is the shortest path while the second inequality means the longest path is the segment from $(0, f(0))$ to $(1, f(0))$ and then going up until $(1, f(1))$.

The intuition from this suggests that the second inequality is equality if and only if $h=0, a . e$., and the first inequality is equality if and only if $h=A$, a.e. The first claim is clear since $\sqrt{1+x^{2}}=1+x$ iff $x=0$.If $h=A$, a.e, then trivially the first inequality holds. Conversely if the first inequality holds, it follows from an examination of the proof of Jensen's inequality that $\phi(A)=\phi(h(x))$, a.e., so $h=A$, a.e. since $\phi$ is injective on $[0, \infty)$.
(7) Optional. Suppose $1<p<\infty, f \in L^{p}=L^{p}((0, \infty))$, relative to Lebesgue measure, and

$$
F(x)=\frac{1}{x} \int_{0}^{x} f(t) d t \quad(0<x<\infty)
$$

(a) Prove Hardy's inequality

$$
\|F\|_{p} \leq \frac{p}{p-1}\|f\|_{p}
$$

which shows that the mapping $f \rightarrow F$ carries $L^{p}$ into $L^{p}$.
(b) Prove that equality holds only if $f=0$ a.e.
(c) Prove that the constant $\frac{p}{p-1}$ cannot be replaced by a smaller one.
(d) If $f>0$ and $f \in L^{1}$, prove that $F \notin L^{1}$.

Suggestions: (a) Assume first that $f \geq 0$ and $f \in C_{c}((0, \infty))$. Integration by parts gives

$$
\int_{0}^{\infty} F^{p}(x) d x=-p \int_{0}^{\infty} F^{p-1}(x) x F^{\prime}(x) d x
$$

Note that $x F^{\prime}=f-F$, and apply Hölder's inequality to $\int F^{p-1} f$. Then derive the general case.
(c) Take $f(x)=x^{-1 / p}$ on $[1, A], f(x)=0$ elsewhere, for large $A$. See also Exercise 14, Chap. 8 in $[\mathrm{R}]$.

Solution. In fact we can show the inequality

$$
\int_{0}^{\infty}|F|^{p} d x \leq \frac{p}{p-1} \int_{0}^{\infty}|f||F|^{p-1} d x
$$

(a) $\vdash\|F\|_{p} \leq \frac{p}{p-1}\|f\|_{p}, f \in \mathcal{L}^{p}(0, \infty), p \in(1, \infty)$

Let $f \in C_{c}(0, \infty), f \geq 0$, first

$$
\begin{aligned}
\int_{0}^{\infty} F^{p}(x) d x & =\left.x F^{p}(x)\right|_{0} ^{\infty}-p \int_{0}^{\infty} F^{p-1} F^{\prime} x d x \\
& =0-p \int_{0}^{\infty} F^{p-1}(f-F) d x
\end{aligned}
$$

so

$$
\begin{equation*}
\int_{0}^{\infty} F^{p}(x) d x=\frac{p}{p-1} \int_{0}^{\infty} F^{p-1} f d x \tag{1}
\end{equation*}
$$

By Hölder's inequality,

$$
\int_{0}^{\infty} F^{p}(x) d x \leq \frac{p}{p-1}\left\{\int_{0}^{\infty} F^{p}(x) d x\right\}^{\frac{1}{q}}\|f\|_{p}
$$

and (a) holds.
Now, for $f \in C_{c}(0, \infty)$, use

$$
|F| \leq \frac{1}{x} \int_{0}^{x}|f|
$$

to get the same inequality.
Finally, for $f \in L^{p}(0, \infty)$, let $f_{n} \in C_{c}(0, \infty), f_{n} \rightarrow f$ in $L^{p}$. Use an approximation argument to show $\left\{F_{n}\right\}$ is Cauchy and tends to $F$ in $\mathcal{L}^{p}$ norm.
(b) $\vdash "="$ hold iff $f=0$ a.e.

Let $f$ satisfy

$$
\|F\|_{p}=\frac{p}{p-1}\|f\|_{p}
$$

If $f$ changes sign,

$$
\begin{gathered}
\widetilde{F}(x)=\frac{1}{x} \int_{0}^{x}|f| d t \\
\|\widetilde{F}\|_{p}>\|F\|_{p}=\frac{p}{p-1}=\||f|\|_{p}
\end{gathered}
$$

Impossible. Therefore $f \geq 0$ say. By an approximation argument one can show that (1) holds for $f \geq$ $0, f \in L^{p}$. Following the proof in (a) one see by the equality condition in Hölder's inequality that $f^{p}=$
const $\left(F^{p-1}\right)^{q}$, which implies there exists some positive constant $c$ such that $F(x)=c f(x)$ a.e. Express this as an ODE for $F$ and and solve it to get $f \equiv 0$ if $f \in L^{p}(0, \infty)$.
(c) Define

$$
f(x)= \begin{cases}x^{-1 / p}, & \text { if } x \in[1, A] \\ 0, & \text { otherwise }\end{cases}
$$

Then $\|f\|_{p}=(\log A)^{1 / p}$ and

$$
F(x)= \begin{cases}0, & \text { if } x \in(0,1) \\ \frac{p}{p-1}\left(x^{-\frac{1}{p}}-x^{-1}\right), & \text { if } x \in[1, A] \\ \frac{p}{p-1}\left(A^{1-\frac{1}{p}}-1\right) x^{-1}, & \text { if } x \in(A, \infty)\end{cases}
$$

Then $\|F\|_{p}^{p}=I_{1}+I_{2}$, where

$$
\begin{aligned}
I_{1} & =\int_{1}^{A}\left(\frac{p}{p-1}\left(x^{-\frac{1}{p}}-x^{-1}\right)\right)^{p} d x \\
& =\left(\frac{p}{p-1}\right)^{p} \int_{1}^{A}\left(x^{-\frac{1}{p}}-x^{-1}\right)^{p} d x \\
I_{2} & =\int_{A}^{\infty}\left(\frac{p}{p-1}\left(A^{1-\frac{1}{p}}-1\right) x^{-1}\right)^{p} d x \\
& =\frac{p^{p}}{(p-1)^{p+1}}\left(1-A^{\frac{1}{p}-1}\right)^{p} d x .
\end{aligned}
$$

Suppose on the contrary that the constant $\frac{p}{p-1}$ can be replaced by $\frac{\gamma p}{p-1}$ for some $\gamma \in(0,1)$. Then there exists $\delta \in(\gamma, 1)$. Note that there exists $A_{0}>1$ such that for $x>A_{0}, x^{-\frac{1}{p}}-x^{-1}>\delta x^{-\frac{1}{p}}$. Then for sufficiently large $A>A_{0}$,

$$
\begin{aligned}
I_{1} & >\frac{\delta p}{p-1} \int_{A_{0}}^{A} x^{-1} d x \\
& =\frac{\delta p}{p-1}\left(\log A-\log A_{0}\right) \\
& >\frac{\gamma p}{p-1} \log A \\
& =\frac{\gamma p}{p-1}\|f\|_{p}^{p}
\end{aligned}
$$

This implies $\|F\|_{p}>\frac{\gamma p}{p-1}\|p\|_{f}$ if $A$ is sufficiently large. Contradiction arises.
(d) Since $f>0$ on $(0, \infty)$, there exists $x_{0}>0$ such that $c_{0}:=\int_{0}^{x_{0}} f(t) d t$. Then

$$
\int_{x_{0}}^{\infty} F(x) d x=\int_{x_{0}}^{\infty} \frac{1}{x} \int_{0}^{x} f(t) d t d x \geq \int_{x_{0}}^{\infty} \frac{1}{x} \int_{0}^{x_{0}} f d t d x \geq \int_{x_{0}}^{\infty} \frac{c_{0}}{x} d x=\infty
$$

showing that $F \notin L^{1}$.

