Solution to MATH5011 homework 8

(1) Let $f, g \in L^p(\mu)$, 1 . Show that the function

$$\Phi(t) = \int_X |f + tg|^p \, d\mu$$

is differentiable at t = 0 and

$$\Phi'(0) = p \int_{Y} |f|^{p-2} fg \, d\mu.$$

Hint: Use the convexity of $t \mapsto |f + tg|^p$ to get

$$|f + tg|^p - |f|^p \le t(|f + g|^p - |f|^p), \quad t > 0$$

and a similar estimate for t < 0.

Solution. Recall that for any convex function φ defined on [0, 1], one has the elementary inequality

$$\frac{\varphi(t) - \varphi(0)}{t - 0} \le \frac{\varphi(1) - \varphi(0)}{1 - 0}, \quad \forall t \in (0, 1),$$

which could be deduced from the definition of convexity. For $p > 1, x \in X$, the function $\varphi(t) = |f(x) + tg(x)|^p$ is differentiable and convex whenever f(x) and g(x) are finite, which can be seen from $\varphi''(t) \ge 0$. Applying the inequality above to this particular convex function, We have

$$\frac{1}{t}\{|f+tg|^p - |f|^p\} \le |f+g|^p - |f|^p, \ \forall t \in (0,1).$$

By replacing t with -t, we obtain a similar inequality

$$|f|^p - |f - g|^p \le \frac{1}{t} \{ |f + tg|^p - |f|^p \}, \ \forall t \in (-1, 0).$$

Now the desired result follows from an application of Lebesgue's dominated convergence theorem.

(2) Suppose f is a measurable function on X, μ is a positive measure on X, and

$$\varphi(p) = \int_X |f|^p d\mu = ||f||_p^p \quad (0$$

Let $E = \{p : \varphi(p) < \infty\}$. Assume $||f||_{\infty} > 0$.

- (a) If $r , <math>r \in E$, and $s \in E$, prove that $p \in E$.
- (b) Prove that $\log \varphi$ is convex in the interior of E and that φ is continuous on E.
- (c) By (a), E is connected. Is E necessarily open? Closed? Can E consist of a single point? Can E be any connected subset of $(0, \infty)$?

- (d) If $r , prove that <math>||f||_p \le \max(||f||_r, ||f||_s)$. Show that this implies the inclusion $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.
- (e) Assume that $\|f\|_r < \infty$ for some $r < \infty$ and prove that

$$||f||_p \to ||f||_\infty$$
 as $p \to \infty$.

Solution.

(a) Write $p = \lambda r + (1 - \lambda)s$ for $0 < \lambda < 1$. By Hölder's inequality,

$$\int_X |f|^p d\mu = \int_X |f|^{\lambda r} |f| (1 - \lambda) s d\mu \le \left(\int_X |f|^r d\mu \right)^{\lambda} \left(\int_X |f|^s d\mu \right)^{1 - \lambda},$$

which shows that φ is finite on [r, z]. It follows that E is an interval.

(b) Rewrite the inequality above as

$$\varphi(\lambda r + (1 - \lambda)s) \le \varphi(r)^{\lambda} \cdot \varphi(s)^{1-\lambda}, \quad (0 < \lambda < 1).$$

It is also true for $\lambda = 0, 1$. Hence for all $\lambda \in [0, 1]$,

$$\log \varphi(\lambda r + (1 - \lambda)s) \le \lambda \log \varphi(r) + (1 - \lambda) \log \varphi(s).$$

since log is increasing. Thus $\log \varphi(p)$ is convex on [r, s]. Hence $\varphi(x)$ is continuous in the interior of E. It follows form monotonicity applying to $\chi_{|f|>1}f$ and $\chi_{|f|\leq 1}f$ that $\varphi(x)$ is also continuous on ∂E .

(c) Let $X = (0, \infty)$ with the Lebesgue measure. E can be any connected subset of $(0, \infty)$. The basic functions to consider are of the form x^k and $x^k |\log x|^m$ near x = 0 and $x = \infty$. Define

$$g_k(x) = x^k \chi_{(0,1/2]}(x),$$

$$h_k(x) = x^k \chi_{(2,\infty)}(x),$$

$$g_{k,m}(x) = x^k |\log x|^m \chi_{(0,1/2]}(x),$$

$$h_{k,m}(x) = x^k |\log x|^m \chi_{(2,\infty)}(x),$$

It is easy to see that $\int_X g_k dx < \infty$ iff k > -1 and $\int_X h_k dx < \infty$ iff k < -1. Since $|\log x| \le C_{\varepsilon} e^{-\varepsilon}$ for $0 \le x \le 1$ and all $\epsilon > 0$, $\int_X g_{k,m} dx$ is finite for k > -1 and infinite for k > -1. For k = -1, direct computations by substituting $u = \log x$ yield

$$\int_X g_{k,m} dx = \int_0^{1/2} x^{-1} |\log x|^m dx = \int_{\log 2}^\infty u^m du,$$

which is finite iff m < -1. Similarly, one can show $\int_X h_{k,m} dx$ is finite for k > -1 and infinite for k > -1.

If k=-1, the integral is finite if and only if m<-1. Note that $g_k^p=g_{pk},g_{k,m}^p=g_{pk,pm}$ and similarly for h. Now for $f=g_{-1,-2}+h_{-1,-2}$, one has E=1. For $E=\emptyset$, take $f=g_{-1}+h_{-1}$. To get $E=(0,\infty)$, one may take $f=e^{-|x|}$. For E=[1,p), take $f=g_{-1/p}+h_{-1,-2}$. Similarly it is easy to see that E can be any connected subset of $(0,\infty)$ for choosing f properly.

- (d) The inequality in (a) implies $\|f\|_p \leq \max(\|f\|_r, \|f\|_s)$. Obviously, if $\|f\|_r < \infty$ and $\|f\|_s < \infty$, then $\|f\|_p < \infty$. Thus $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.
- (e) Denote $E_a:=\{x:a\leq |f(x)|\}$ for every $0< a<\|f\|_{\infty}$, then $0<\mu(E_a)<\infty$. $(\|f\|_r<\infty$ implies $\mu(E_a)<\infty$.) Thus

$$||f||_p = \left(\int_X |f|^p d\mu\right)^{1/p} \ge \left(\int_{E_a} |f|^p d\mu\right)^{1/p} \ge a(\mu(E_a))^{1/p},$$

which implies $\underline{\lim}_{p\to\infty}\|f\|_p\geq a$. Since a is arbitrary, we have $\underline{\lim}_{p\to\infty}\|f\|_p\geq \|f\|_\infty$. On the other hand, for p>r,

$$||f||_p = \left(\int_X |f|^{p-r}|f|r\,d\mu\right)^{1/p} \le ||f||_r^{r/p} \, ||f||_\infty^{1-r/p},$$

which implies $\overline{\lim}_{p\to\infty}\|f\|_p\leq \|f\|_\infty$. In conclusion, we have

$$\lim_{p \to \infty} ||f||_{\infty} = ||f||_{\infty}.$$

(3) Assume, in addition to the hypothesis of the last problem, that

$$\mu(X) = 1.$$

- (a) Prove that $\|f\|_r \le \|f\|_s$ if $0 < r < s \le \infty$.
- (b) Under what conditions does it happen that $0 < r < s \le \infty$ and $\|f\|_r = \|f\|_s < \infty$?
- (c) Prove that $L^r(\mu) \supset L^s(\mu)$ if 0 < r < s. Under what conditions do these two spaces contain the same functions?
- (d) Assume that $||f||_r < \infty$ for some r > 0, and prove that

$$\lim_{p\to 0}\|f\|_p=\exp\left\{\int_X\!\log|f|\,d\mu\right\}$$

if $\exp\{-\infty\}$ is defined to be 0.

Solution.

(a) If $s < \infty$, the conclusion from Hölder's inequality,

$$\int_X |f|^r \, d\mu \le \left(\int_X |f|^s \, d\mu \right)^{r/s} \left(\int_X 1 \, d\mu \right)^{1 - r/s} = \|f\|_s^r \, .$$

If $s = \infty$, the desired result follows from

$$||f||_r \le ||f||_{\infty} \left(\int_X 1 \, d\mu \right)^{1/r} = ||f||_{\infty}.$$

- (b) From the equality sign characterization in the Hölder inequality it is easy to see that $||f||_r = ||f||_s < \infty$ if and only if $|f| = ||f||_{\infty} < \infty$ a.e..
- (c) We claim that under the condition $\mu(X) < \infty$, $L^r(\mu) = L^s(\mu)$ for $0 < r < s \le \infty$ if and only if the following property (call it L) holds:

There exists $\varepsilon_0 > 0$ such that for any measurable set $E \subset X$ with $\mu(E) > 0$ we have $\mu(E) > \varepsilon_0$.

In fact, if Property L holds, let $f \in L^r(\mu)$ and denote $E_n := \{x : |f| \ge n\}$. Then there exists $n_0 \in \mathbb{N}$ such that $\mu(E_{n_0}) = 0$ and thus $f \in L^{\infty}(\mu)$. Otherwise for all n, $\mu(E_n) > 0$. Thus $\mu(\{x : |f(x)| = \infty\}) \ge \lim_{n \to \infty} \mu(E_n) \ge \varepsilon_0$ and then $||f||_r = \infty$, a contradiction.

Conversely, suppose there is a sequence of measurable sets $\{E_n\}$ with $0 < \mu(E_n) < 3^{-n}$. Without loss of generality, E_n are mutually disjoint. Denote $a_n := \mu(E_n)$ and define

$$f = \begin{cases} \sum_{n=1}^{\infty} a_n^{-1/s} \chi_{E_n}, & \text{if } s < \infty, \\ \sum_{n=1}^{\infty} a_n^{-\frac{1}{2r}} \chi_{E_n}, & \text{if } s = \infty. \end{cases}$$

Then $f \in L^r$ but $f \notin L^s$. The proof is completed.

(d) Note $x - 1 - \log x \ge 0$ on $[0, \infty)$ implies that

$$\int_{\{|f|>1\}} \log |f| d\mu < \infty.$$

If $\mu(\{|f|=0\}) > 0$, it suffices to prove the equality by showing $\lim_{p\to 0} ||f||_p = 0$. There is a small s > 1, with s' be its conjugate s.t.

$$||f||_{p} = \left\{ \int_{X} |f|^{p} \chi_{\{|f|>0\}} d\mu \right\}^{\frac{1}{p}}$$

$$\leq (\mu\{|f|>0\})^{\frac{1}{s'p}} ||f||_{sp} \text{ by H\"older inequality}$$

$$\leq (\mu\{|f|>0\})^{\frac{1}{s'p}} ||f||_{r} \to 0 \text{ as } p \to 0$$

We may suppose $\infty > |f| > 0$ a.e. By Jensen's inequality, we have

$$\log ||f||_p = \frac{1}{p} \log \int_X |f|^p \, d\mu \ge \frac{1}{p} \int_X \log |f|^p \, d\mu = \int_X \log |f| \, d\mu.$$

On the other hand, $x - 1 - \log x \ge 0$ on $[0, \infty)$ implies $\frac{\|f\|_p^p - 1}{p} \ge \log \|f\|_p$. Thus

$$\int_X \log|f| \, d\mu \le \log \|f\|_p \le \int_X \frac{|f|^p - 1}{p} \, d\mu$$

since $\mu(X)=1$. Note that by convexity of the map $p\mapsto |f|^p$ we have $\frac{|f|^p-1}{p}$ is increasing in p, which implies $\frac{|f|^p-1}{p}\le \frac{|f|^r-1}{r}\in L^1(\mu)$ and $\lim_{p\to 0}\frac{|f|^p-1}{p}=\log|f|$. By Lebesgue's dominated convergence theorem for |f|>1 and monotone convergence theorem for |f|<1, we have

$$\lim_{p \to 0} \int_X \frac{|f|^p - 1}{p} \, d\mu = \lim_{p \to 0} \int_{\{|f| > 1\}} \frac{|f|^p - 1}{p} \, d\mu + \lim_{p \to 0} \int_{\{|f| < 1\}} \frac{|f|^p - 1}{p} \, d\mu = \int_X \log|f| \, d\mu.$$

Thus by sandwich rule

$$\lim_{p \to 0} \|f\|_p = \exp\left\{ \int_X \log|f| \, d\mu \right\}$$

(4) For some measures, the relation r < s implies $L^r(\mu) \subset L^s(\mu)$; for others, the inclusion is reversed; and there are some for which $L^r(\mu)$ does not contain $L^s(\mu)$ is $r \neq s$. Give examples of these situations, and find conditions on μ under which these situations will occur.

Solution.

First, we give examples of these situations:

- (a) For X = [0, 1] with usual Lebesgue measure, we have $L^r(\mu) \supset L^s(\mu)$ if r < s.
- (b) For $X = \mathbb{N}$ with counting measure, we have $L^r(\mu) \subset L^s(\mu)$ if r < s.
- (c) For $X = \mathbb{R}$ with usual Lebesgue measure, we have $L^r(\mu) \neq L^s(\mu)$ if $r \neq s$.

Second, we give simple conditions on μ under which these situations occur correspondingly:

- (a) $\mu(X) < \infty$.
- (b) Property L in 6(c) holds.
- (c) $\mu(X) = \infty$ and Property L in 6(c) fails to hold.
- (5) Suppose $\mu(\Omega) = 1$, and suppose f and g are positive measurable functions on Ω such that $fg \geq 1$. Prove that

$$\int_{\Omega} f \, d\mu \cdot \int_{\Omega} g \, d\mu \ge 1.$$

Solution. Since $fg \ge 1$, we have $\sqrt{fg} \ge 1$ and so by Hölder's inequality,

$$1 \le \int_{\Omega} \sqrt{f} \sqrt{g} \, d\mu \le \left(\int_{\Omega} f \, d\mu \right)^{1/2} \left(\int_{\Omega} g \, d\mu \right)^{1/2} = \left(\int_{\Omega} f \, d\mu \cdot \int_{\Omega} g \, d\mu \right)^{1/2}.$$

(6) Suppose $\mu(\Omega) = 1$ and $h: \Omega \to [0, \infty]$ is measurable. If

$$A = \int_{\Omega} h \, d\mu,$$

prove that

$$\sqrt{1+A^2} \le \int_{\Omega} \sqrt{1+h^2} \, d\mu \le 1+A.$$

If μ is Lebesgue measure on [0, 1] and if h is continuous, h = f', the above inequalities have a simple geometric interpretation. From this, conjecture (for general Ω) under what conditions on h equality can hold in either of the above inequalities, and prove your conjecture.

Solution. The function $\phi(x) = \sqrt{1+x^2}$ is convex since its second derivative is always positive. Hence the first inequality follows from Jensen's inequality. The second equality follows from $|\Omega| = 1$ and $\sqrt{1+x^2} \le 1+x$ for all $x \ge 0$.

In the case that $\Omega = [0,1]$ with μ the Lebesgue measure and h = f' is continuous, then $\int_0^1 \sqrt{1 + (f')^2} dx$ is the arc length of the graph of f. Then A = f(1) - f(0). The first inequality means that the straight line is the shortest path while the second inequality means the longest path is the segment from (0, f(0)) to (1, f(0)) and then going up until (1, f(1)).

The intuition from this suggests that the second inequality is equality if and only if h = 0, a.e., and the first inequality is equality if and only if h = A, a.e. The first claim is clear since $\sqrt{1 + x^2} = 1 + x$ iff x = 0. If h = A, a.e., then trivially the first inequality holds. Conversely if the first inequality holds, it follows from an examination of the proof of Jensen's inequality that $\phi(A) = \phi(h(x)), a.e.$, so h = A, a.e. since ϕ is injective on $[0, \infty)$.

(7) Optional. Suppose 1 , relative to Lebesgue measure, and

$$F(x) = \frac{1}{x} \int_0^x f(t) dt$$
 $(0 < x < \infty).$

(a) Prove Hardy's inequality

$$||F||_p \le \frac{p}{p-1} ||f||_p$$

which shows that the mapping $f \to F$ carries L^p into L^p .

- (b) Prove that equality holds only if f = 0 a.e.
- (c) Prove that the constant $\frac{p}{p-1}$ cannot be replaced by a smaller one.
- (d) If f > 0 and $f \in L^1$, prove that $F \notin L^1$.

Suggestions: (a) Assume first that $f \geq 0$ and $f \in C_c((0,\infty))$. Integration by parts gives

$$\int_0^\infty F^p(x) dx = -p \int_0^\infty F^{p-1}(x) x F'(x) dx.$$

Note that xF' = f - F, and apply Hölder's inequality to $\int F^{p-1}f$. Then derive the general case.

(c) Take $f(x) = x^{-1/p}$ on [1, A], f(x) = 0 elsewhere, for large A. See also Exercise 14, Chap. 8 in [R].

Solution. In fact we can show the inequality

$$\int_0^\infty |F|^p \, dx \le \frac{p}{p-1} \int_0^\infty |f| |F|^{p-1} \, dx.$$

(a)
$$\vdash ||F||_p \le \frac{p}{p-1} ||f||_p, f \in \mathcal{L}^p(0,\infty), p \in (1,\infty)$$

Let $f \in C_c(0, \infty), f \ge 0$, first

$$\int_0^\infty F^p(x)dx = xF^p(x)\Big|_0^\infty - p \int_0^\infty F^{p-1}F'xdx$$
$$= 0 - p \int_0^\infty F^{p-1}(f - F)dx,$$

so

$$\int_0^\infty F^p(x)dx = \frac{p}{p-1} \int_0^\infty F^{p-1}fdx. \tag{1}$$

By Hölder's inequality,

$$\int_0^\infty F^p(x)dx \le \frac{p}{p-1} \left\{ \int_0^\infty F^p(x)dx \right\}^{\frac{1}{q}} \|f\|_p$$

and (a) holds.

Now, for $f \in C_c(0, \infty)$, use

$$|F| \leq \frac{1}{x} \int_0^x |f|$$

to get the same inequality.

Finally, for $f \in L^p(0,\infty)$, let $f_n \in C_c(0,\infty)$, $f_n \to f$ in L^p . Use an approximation argument to show $\{F_n\}$ is Cauchy and tends to F in \mathcal{L}^p norm.

(b) \vdash " = " hold iff f = 0 a.e.

Let f satisfy

$$||F||_p = \frac{p}{p-1}||f||_p.$$

If f changes sign,

$$\widetilde{F}(x) = \frac{1}{x} \int_0^x |f| dt$$

$$\|\widetilde{F}\|_p > \|F\|_p = \frac{p}{p-1} = \||f|\|_p$$

Impossible. Therefore $f \ge 0$ say. By an approximation argument one can show that (1) holds for $f \ge 0$, $f \in L^p$. Following the proof in (a) one see by the equality condition in Hölder's inequality that $f^p = 0$

const $(F^{p-1})^q$, which implies there exists some positive constant c such that F(x) = cf(x) a.e. Express this as an ODE for F and and solve it to get $f \equiv 0$ if $f \in L^p(0, \infty)$.

(c) Define

$$f(x) = \begin{cases} x^{-1/p}, & \text{if } x \in [1, A], \\ 0, & \text{otherwise.} \end{cases}$$

Then $||f||_p = (\log A)^{1/p}$ and

$$F(x) = \begin{cases} 0, & \text{if } x \in (0,1), \\ \frac{p}{p-1} \left(x^{-\frac{1}{p}} - x^{-1} \right), & \text{if } x \in [1,A], \\ \frac{p}{p-1} \left(A^{1-\frac{1}{p}} - 1 \right) x^{-1}, & \text{if } x \in (A,\infty). \end{cases}$$

Then $||F||_p^p = I_1 + I_2$, where

$$I_{1} = \int_{1}^{A} \left(\frac{p}{p-1} \left(x^{-\frac{1}{p}} - x^{-1}\right)\right)^{p} dx$$

$$= \left(\frac{p}{p-1}\right)^{p} \int_{1}^{A} \left(x^{-\frac{1}{p}} - x^{-1}\right)^{p} dx$$

$$I_{2} = \int_{A}^{\infty} \left(\frac{p}{p-1} \left(A^{1-\frac{1}{p}} - 1\right) x^{-1}\right)^{p} dx$$

$$= \frac{p^{p}}{(p-1)^{p+1}} \left(1 - A^{\frac{1}{p}-1}\right)^{p} dx.$$

Suppose on the contrary that the constant $\frac{p}{p-1}$ can be replaced by $\frac{\gamma p}{p-1}$ for some $\gamma \in (0,1)$. Then there exists $\delta \in (\gamma,1)$. Note that there exists $A_0 > 1$ such that for $x > A_0$, $x^{-\frac{1}{p}} - x^{-1} > \delta x^{-\frac{1}{p}}$. Then for sufficiently large $A > A_0$,

$$I_1 > \frac{\delta p}{p-1} \int_{A_0}^A x^{-1} dx$$

$$= \frac{\delta p}{p-1} (\log A - \log A_0)$$

$$> \frac{\gamma p}{p-1} \log A$$

$$= \frac{\gamma p}{p-1} \|f\|_p^p.$$

This implies $||F||_p > \frac{\gamma p}{p-1} ||p||_f$ if A is sufficiently large. Contradiction arises.

(d) Since f > 0 on $(0, \infty)$, there exists $x_0 > 0$ such that $c_0 := \int_0^{x_0} f(t) dt$. Then

$$\int_{x_0}^{\infty} F(x) \, dx = \int_{x_0}^{\infty} \frac{1}{x} \int_{0}^{x} f(t) \, dt dx \ge \int_{x_0}^{\infty} \frac{1}{x} \int_{0}^{x_0} f \, dt dx \ge \int_{x_0}^{\infty} \frac{c_0}{x} \, dx = \infty,$$

showing that $F \notin L^1$.