MATH4240 Tutorial 6

2024-02-26

Example for transition matrix in the long-time limit

Recall that the following proposition from lecture:

Theorem 1. *If P is the transition matrix of a (finite) Markov chain, and*

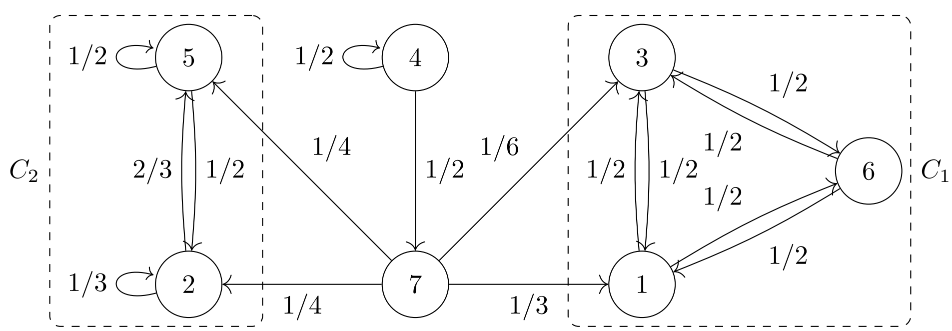
- 1 *is a simple eigenvalue of* P
- *there exists a left eigenvector* π *of* P *corresponding to* 1 *having nonnegative entries*
- *all other (complex) eigenvalues of* P *have moduli less than* 1

then $\lim_{n\to\infty} P^n = (\pi, \pi, \dots, \pi)^\mathsf{T}$, *assuming* π *is normalized to have* $\sum \pi_i = 1$

Consider again the Markov chain with transition matrix from the previous tutorial

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \!\!\! \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 0 & 0 & 2/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/3 & 1/4 & 1/6 & 0 & 1/4 & 0 & 0 \end{bmatrix} \end{array}$$

with transition diagram



We want to compute $\lim_{n\to\infty} P^n$.

Let us first rewrite the matrix in canonical form[1] by grouping the states together

$$P_{\text{canonical}} = \begin{array}{c} \\ 1 \\ 3 \\ 6 \\ 2 \\ 5 \\ 4 \\ 7 \end{array} \!\!\! \begin{array}{c} \begin{array}{ccccccc} 1 & 3 & 6 & 2 & 5 & 4 & 7 \end{array} \\ \left[\begin{array}{ccc|cc|cc} 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 1/3 & 1/6 & 0 & 1/4 & 1/4 & 0 & 0 \end{array}\right] \end{array} = \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ S_1 & S_2 & Q \end{pmatrix}$$

[1] I don't think there is a standard notation for canonical form, so I will just use whatever is convenient.

here the states are arranged such that whenever i is listed before j, state i does not lead to state j unless they are in the same irreducible closed set or are both transient, so $P_{\text{canonical}}$ is upper triangular block matrix. We will work exclusively on the fundamental form and only convert back to the original form at the end.

Recall from last tutorial session, the limit transition matrix $\lim_{n\to\infty} \tilde{P}^n$ on the absorbing chain

$$\tilde{P} = \begin{array}{c} \\ C_1 \\ C_2 \\ 4 \\ 7 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} C_1 & C_2 & 4 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} \end{array} = \begin{pmatrix} I & 0 \\ S & Q \end{pmatrix} \quad \text{is} \quad \lim_{n\to\infty} \tilde{P}^n = \begin{pmatrix} I & 0 \\ NS & 0 \end{pmatrix} = \begin{array}{c} \\ C_1 \\ C_2 \\ 4 \\ 7 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} C_1 & C_2 & 4 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} \end{array}$$

with its *fundamental matrix*[2] $N = (I-Q)^{-1} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ and $NS = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. So to find $\lim_{n\to\infty} P^n$ it remains to find the limit transition matrices on the two irreducible closed sets.

For $C_1 = \{1,3,6\}$, the eigenvalues of the transition matrix P_1 are $1, -1/2, -1/2$. The left eigenvector of 1 satisfies

$$\pi_1 = \pi_1 P_1 = \pi_1 \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \quad \text{which gives} \quad \pi_1 = c(1,1,1),\ c \in \mathbb{R}$$

So 1 is a simple eigenvalue having a (left) eigenvector with nonnegative entries, and all other eigenvalues have modulus less than 1. By the proposition in lecture (Theorem 1), $\pi_1 = (1/3, 1/3, 1/3)$ and $\lim_{n\to\infty} P_1^n = (\pi_1, \pi_1, \pi_1)^\mathsf{T}$.

For $C_2 = \{2,5\}$, the eigenvalues of the transition matrix P_2 are $1, -1/6$. The left eigenvector of 1 satisfies

$$\pi_2 = \pi_2 P_2 = \pi_2 \begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix} \quad \text{which gives} \quad \pi_2 = c(3,4),\ c \in \mathbb{R}$$

So 1 is a simple eigenvalue having a (left) eigenvector with nonnegative entries, and all other eigenvalues have modulus less than 1. By Theorem 1, $\pi_2 = (3/7, 4/7)$ and $\lim_{n\to\infty} P_2^n = (\pi_2, \pi_2)^\mathsf{T}$.

As mentioned in lecture, the bottom-left block of $\lim_{n\to\infty} P_{\text{can}}^n$ is $((\rho_{C_1}(x_i)\pi_1), (\rho_{C_2}(x_i)\pi_2), \ldots)$, so combined this gives (while *severely* abusing notation)

$$\begin{aligned}
\lim_{n\to\infty} P_{\text{can}}^n &= \begin{pmatrix} \begin{pmatrix} \vec{1}_3 & 0 \\ 0 & \vec{1}_2 \end{pmatrix} & 0 \\ NS & 0 \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} \vec{1}_3\pi_1 & 0 \\ 0 & \vec{1}_2\pi_2 \end{pmatrix} & 0 \\ NS\begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix} & 0 \end{pmatrix} = \begin{pmatrix} \begin{matrix} - & \pi_1 & - \\ - & \pi_1 & - \\ - & \pi_1 & - \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} - & \pi_2 & - \\ - & \pi_2 & - \end{matrix} & 0 \\ \begin{matrix} - & (1/2)\pi_1 & - \\ - & (1/2)\pi_1 & - \end{matrix} & \begin{matrix} - & (1/2)\pi_2 & - \\ - & (1/2)\pi_2 & - \end{matrix} & 0 \end{pmatrix} \\
&= \begin{array}{c} \\ 1 \\ 3 \\ 6 \\ 2 \\ 5 \\ 4 \\ 7 \end{array}\!\!\begin{array}{c} \begin{array}{ccc|cc|cc} 1 & 3 & 6 & 2 & 5 & 4 & 7 \end{array} \\ \left[\begin{array}{ccc|cc|cc} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 3/7 & 4/7 & 0 & 0 \\ 0 & 0 & 0 & 3/7 & 4/7 & 0 & 0 \\ \hline 1/6 & 1/6 & 1/6 & 3/14 & 2/7 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 3/14 & 2/7 & 0 & 0 \end{array}\right] \end{array}
\end{aligned}$$

[2] Here we only define fundamental matrix if every recurrent state is absorbing (*absorbing* chain); generalizations exist but are beyond our scope.

with $\vec{1}_n$ being the column vector with n 1s. Permuting the states back[3], we obtain

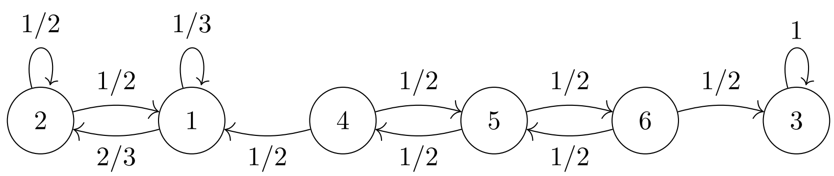
$$\lim_{n\to\infty} P^n = \begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6\\7 \end{array} & \left[\begin{array}{ccccccc} 1/3 & 0 & 1/3 & 0 & 0 & 1/3 & 0 \\ 0 & 3/7 & 0 & 0 & 4/7 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 0 & 1/3 & 0 \\ 1/6 & 3/14 & 1/6 & 0 & 2/7 & 1/6 & 0 \\ 0 & 3/7 & 0 & 0 & 4/7 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 0 & 1/3 & 0 \\ 1/6 & 3/14 & 1/6 & 0 & 2/7 & 1/6 & 0 \end{array}\right] \end{array}$$

Example from lecture note

Let us look at the transition matrix from lecture note p.22 (181/323):

$$P = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} & \left[\begin{array}{cccccc} 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \end{array}\right] \end{array}$$

with transition diagram



Note that P is already in canonical form

$$P_{\text{can}} = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} & \left[\begin{array}{cc|c|ccc} 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \end{array}\right] \end{array} = \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ S_1 & S_2 & Q \end{pmatrix}$$

on which the corresponding transition matrix on the absorbing chain (which is a gambler's ruin chain after a renaming) is

$$\tilde{P} = \begin{array}{cc} & \begin{array}{ccccc} \{1,2\} & \{3\} & 4 & 5 & 6 \end{array} \\ \begin{array}{c} \{1,2\}\\ \{3\}\\4\\5\\6 \end{array} & \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \end{array}\right] \end{array} = \begin{pmatrix} I_2 & 0 \\ S & Q \end{pmatrix}$$

Again its fundamental matrix is $N = (I-Q)^{-1} = \begin{pmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{pmatrix}$ and $NS = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}$

[3] It is necessary to permute the states back as by definition $P_{\text{can}} = CPC^{-1}$ for some permutation matrix C and so $\lim_{n\to\infty} P_{\text{can}}^n = C(\lim_{n\to\infty} P^n)C^{-1}$ differs from $\lim_{n\to\infty} P^n$ by a similar transformation (unless $P = P_{\text{can}}$ is already in canonical form).

The eigenvalues of the block transition matrices are $1, -1/6$ and 1 respectively, which we can verify that the condition of the proposition holds, and from solving the equations $\pi = \pi P$ the stationary distributions on the blocks are respectively

$$\begin{aligned} \pi_1 &= (3/7, 4/7) \\ \pi_2 &= (1) \end{aligned}$$

Hence (again abusing notation)

$$\lim_{n\to\infty} P^n = \lim_{n\to\infty} P^n_{\text{can}} = \begin{pmatrix} \begin{pmatrix} \vec{1}_2 & 0 \\ 0 & \vec{1}_1 \end{pmatrix} & 0 \\ NS & 0 \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix} = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 3/7 & 4/7 & 0 & 0 & 0 & 0 \\ 2 & 3/7 & 4/7 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 4 & 9/28 & 3/7 & 1/4 & 0 & 0 & 0 \\ 5 & 3/14 & 2/7 & 1/2 & 0 & 0 & 0 \\ 6 & 3/28 & 3/7 & 3/4 & 0 & 0 & 0 \end{array}$$

Extra: Some trick

Applying of Perron–Frobenius theorem on (right) stochastic matrices, we obtain

Theorem 2. *Let P be the transition matrix of some finite Markov chain. Suppose for some $n \geq 1$, all entries of P^n are positive, then the conditions to Theorem 1 are satisfied.*

Such transition matrix is said to be *primitive*, and the corresponding chain *regular*.

Note that if P^n only has positive entries, then so is P^{n+1} (try to prove it!). So to show that Theorem 1 holds on the transition matrix P of some *irreducible* Markov chain, we can

- Compute all eigenvalues of P, then find all solutions of $\pi = \pi P$ (which is more or less inevitable); or
- Keep raising powers and see if some P^n has only positive entries[4]

For example, for $P_1 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$ from the first example, all entries of $P_1^2 = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$ are positive, so the proposition holds on P_1. This is sometimes (a bit) faster than computing moduli of all eigenvalues of P_1.

Extra: Proof of limit transition matrix

It appears that the lecture note did not give how the limit of the transition matrix (in canonical form) is obtained. It suffices to consider the bottom-left part.

Theorem 3. *Suppose $C \subseteq S$ is a finite irreducible closed set on which the limit transition matrix takes the form $\lim_{n\to\infty} P^n_C = (\pi, \pi, \ldots, \pi)^\mathsf{T}$ with π being the (unique) stationary distribution on C. Then for all $x \in S_T$ and $y \in C$, $\lim_{n\to\infty} P^n(x, y) = \rho_C(x)\pi_y$*

Proof. Let $T = \min\{\, k \geq 1 \mid X_k \in C \,\}$. Then for $n \geq 1$, as C is finite and irreducible,

$$\begin{aligned} P^n(x,y) = P_x(X_n = y) &= \sum_{i=1}^{n} \sum_{z\in C} P_x(X_n = y,\, T = i,\, X_i = z) \\ &= \sum_{z\in C} \sum_{i=1}^{n} P_x(T = i,\, X_i = z) P^{n-i}(z, y) \\ &= \sum_{z\in C} \sum_{i=1}^{\infty} P_x(T = i,\, X_i = z) P^{n-i}(z, y) \chi_{i\leq n} \end{aligned}$$

[4] If the chain has N states, it suffices to look at $n \geq (N-1)^2 + 1$ (try to prove it!).

Noting that $P^{n-i}(z,y)\chi_{i\leq n} \xrightarrow{n\to\infty} \pi_y$ and $P_x(T = i,\, X_i = z)P^{n-i}(z,y)\chi_{i\leq n} \leq P_x(T = i)$ with which $\sum_{i=1}^{\infty} P_x(T = i) = \rho_C(x) < \infty$ as C is closed, by dominated convergence theorem we have

$$\begin{aligned}
\lim_{n\to\infty} P^n(x,y) &= \sum_{z\in C}\sum_{i=1}^{\infty} P_x(T = i,\, X_i = z) \lim_{n\to\infty} P^{n-i}(z,y)\chi_{i\leq n} \\
&= \sum_{z\in C}\sum_{i=1}^{\infty} P_x(T = i,\, X_i = z)\pi_y \\
&= \pi_y \sum_{i=1}^{\infty}\sum_{z\in C} P_x(T = i,\, X_i = z) \\
&= \pi_y \rho_C(x)
\end{aligned}$$

□