MATH4240 Tutorial 5

2024-02-19

Example for previous topic

This is an example that was supposed to be in the last tutorial session (that I forgot to include in).

Recall that we can decompose the state space of a (finite) Markov chain into irreducible closed sets $C_1, C_2, \dots$ and transient part $\mathscr{S}_T$, which gives us a transition matrix of the block form

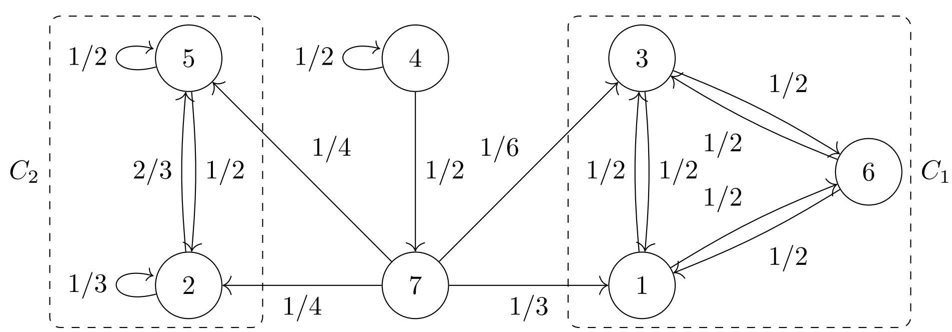
$$P = \begin{array}{c} \\ C_1 \\ C_2 \\ \vdots \\ C_k \\ \mathscr{S}_T \end{array} \begin{array}{c} \begin{array}{ccccc} C_1 & C_2 & \dots & C_k & \mathscr{S}_T \end{array} \\ \begin{bmatrix} P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_k & 0 \\ Q_1 & Q_2 & \dots & Q_k & Q \end{bmatrix} \end{array}$$

From last tutorial we know that $I - Q$ is invertible, and $Q^n \xrightarrow{n\to\infty} 0$.

Consider a Markov chain with transition matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 0 & 0 & 2/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/3 & 1/4 & 1/6 & 0 & 1/4 & 0 & 0 \end{bmatrix} \end{array}$$

from which we can draw the transition diagram



We can see that $C_1 = \{1, 3, 6\}$, $C_2 = \{2, 5\}$ are irreducible closed, and $\mathscr{S}_T = \{4, 7\}$.

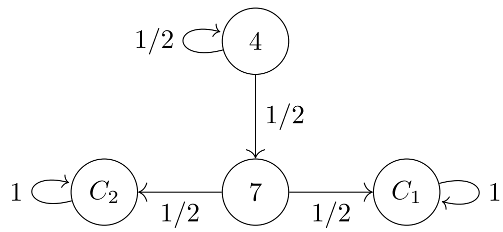
We can write the transition matrix as

$$P = \begin{array}{c} \\ 1 \\ 3 \\ 6 \\ 2 \\ 5 \\ 4 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccc|cc|cc} 1 & 3 & 6 & 2 & 5 & 4 & 7 \end{array} \\ \left[\begin{array}{ccc|cc|cc} 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 1/3 & 1/6 & 0 & 1/4 & 1/4 & 0 & 0 \end{array}\right] \end{array}$$

and further write it as

$$\tilde{P} = \begin{array}{c} \\ C_1 \\ C_2 \\ 4 \\ 7 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} C_1 & C_2 & 4 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} I_2 & \mathbf{0} \\ S & Q \end{bmatrix}$$

with $S = \begin{pmatrix} 0 & 0 \\ 1/2 & 1/2 \end{pmatrix}$, $Q = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 0 \end{pmatrix}$ as $\tilde{P}(7, C_1) = \sum_{x \in C_1} P(7, x) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$ and similarly $\tilde{P}(7, C_2) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. This effectively contracts the original chain to a chain on states $C_1, C_2, 4, 7$ where every recurrent state is absorbing (and thus is an absorbing chain):



By one-step argument,

$$\begin{aligned} \rho_{4,C_1} &= \rho_{C_1}(4) = \tilde{P}(4,4)\rho_{C_1}(4) + \tilde{P}(4,7)\rho_{C_1}(7) \\ \rho_{7,C_1} &= \rho_{C_1}(7) = \tilde{P}(7, C_1) \\ \rho_{4,C_2} &= \rho_{C_2}(4) = \tilde{P}(4,4)\rho_{C_2}(4) + \tilde{P}(4,7)\rho_{C_2}(7) \\ \rho_{7,C_2} &= \rho_{C_2}(7) = \tilde{P}(7, C_2) \end{aligned}$$

While this system can be easily solved[1], let us rewrite it in the following form

$$\begin{pmatrix} \rho_{C_1}(4) & \rho_{C_2}(4) \\ \rho_{C_1}(7) & \rho_{C_2}(7) \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_{C_1}(4) & \rho_{C_2}(4) \\ \rho_{C_1}(7) & \rho_{C_2}(7) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1/2 & 1/2 \end{pmatrix}$$
$$\text{or} \quad A = QA + S$$

with $A = \begin{pmatrix} \rho_{C_1}(4) & \rho_{C_2}(4) \\ \rho_{C_1}(7) & \rho_{C_2}(7) \end{pmatrix}$. As $I - Q$ is invertible with inverse $N = (I - Q)^{-1}$, this means that

$$A = NS = \begin{pmatrix} 1/2 & -1/2 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

Also, from the previous tutorial

$$\lim_{n\to\infty} \tilde{P}^n = \begin{bmatrix} I & 0 \\ NS & 0 \end{bmatrix} = \begin{array}{c} \\ C_1 \\ C_2 \\ 4 \\ 7 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} C_1 & C_2 & 4 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} \end{array}$$

so A is the bottom-left block of $\lim_{n\to\infty} \tilde{P}^n$. Note also that $N_{ij} = (I - Q)^{-1}_{ij} = \sum_{k=0}^{\infty} Q^k_{ij} = \sum_{k=0}^{\infty} P^k(i, j) = E_i(N(j))$ (with appropriate indexing; see also the previous tutorial)

[1] See also HW3 Q20(b)

Birth and death chain

Recall that for an irreducible birth and death chain on nonnegative integers $\mathbb{N}$ with $p_x = P(x, x+1)$ and $q_x = P(x, x-1)$, various properties depend on $\gamma_y = \prod_{x=1}^{y} q_x/p_x$ (with $\gamma_0 = 1$):

- $P_x(T_a < T_b) = (\sum_{y=x}^{b-1} \gamma_y)/(\sum_{y=a}^{b-1} \gamma_y)$ for $a < x < b$
- $P_x(T_a > T_b) = (\sum_{y=a}^{x-1} \gamma_y)/(\sum_{y=a}^{b-1} \gamma_y)$ for $a < x < b$
- (HW3 Q26; also Q28) if $\sum_{y=0}^{\infty} \gamma_y = \infty$, then $\rho_{x0} = 1$ for all $x \geq 1$, and the chain is recurrent; otherwise $\rho_{x0} = (\sum_{y=x}^{\infty} \gamma_y)/(\sum_{y=0}^{\infty} \gamma_y)$ on $x \geq 1$, and the chain is transient

As an application of these properties, let us go through Q30 in HW3.

HW3 Q30

Consider a birth and death chain on nonnegative integers with $p_x = \frac{x+2}{2(x+1)}$ and $q_x = \frac{x}{2(x-1)}$ on $x \geq 0$. We want to compute

- $P_x(T_a < T_b)$ for $a < x < b$
- ρ_{x0} for $x > 0$

By definition, $\frac{q_x}{p_x} = \frac{x}{x+2}$ on $x \geq 1$, so on $y \geq 1$

$$\gamma_y = \prod_{x=1}^{y} \frac{q_x}{p_x} = \frac{1}{3}\frac{2}{4}\cdots\frac{x-1}{x+1}\frac{x}{x+2} = \frac{2}{(x+1)(x+2)} = \frac{2}{x+1} - \frac{2}{x+2}$$

so on $c < d$

$$\sum_{y=c}^{d-1} \gamma_y = \frac{2}{c+1} - \frac{2}{d+1}$$

and

$$\sum_{y=0}^{\infty} \gamma_y = 2 < \infty$$

which implies the chain is transient and

- $$P_x(T_a < T_b) = \left(\sum_{y=x}^{b-1} \gamma_y\right) / \left(\sum_{y=a}^{b-1} \gamma_y\right) = \left(\frac{2}{x+1} - \frac{2}{b-1}\right) / \left(\frac{2}{a+1} - \frac{2}{b+1}\right) = \frac{(b-x)(a+1)}{(b-a)(x+1)}$$

- $$\rho_{x0} = \left(\sum_{y=x}^{\infty} \gamma_y\right) / \left(\sum_{y=0}^{\infty} \gamma_y\right) = \frac{2}{x+1}/2 = \frac{1}{x+1}$$

Branching chain

Recall that in a branching chain, its behavior can be studied with the generating function[2] $\Phi(t) = \sum_{k=0}^{\infty} p_k t^k$ of the distribution of the offspring produced by each individual:

- The expected number of offspring of an individual in one generation is $\mu = E_1(X_1) = \Phi'(1)$

[2] Technically speaking this differs from the (usual) moment generating function of the probability distribution by a change of variable.

- (HW3 Q35) $E_x(X_n) = x\mu^n$
- $P(1,0) = \Phi(0)$
- The extinction probability $\rho = \rho_{10} \in [0,1]$ with $\rho_{x0} = \rho^x$ is the limit of the recurrence x_n with $x_0 = 0$ and $x_{n+1} = \Phi(x_n)$, and thus satisfies $\rho = \Phi(\rho)$, which except in the degenerate case $p_1 = 1$ and depending on μ there are at most 2 solutions in $[0,1]$, one of which is $t = 1$

As an example, let us go through HW3 Q33.

HW3 Q33

Consider a branching chain with $f(0) = f(3) = 1/2$. We want to compute the extinction probability ρ.

The generating function is $\Phi(t) = f(0) + f(3)t^3 = (1+t^3)/2$ with $\mu = \Phi'(1) = \frac{3}{2} > 1$. This means that $\Phi(t) = t$ has more than 1 solutions and ρ is the smaller one in $[0,1)$. Knowing that $t = 1$ is always a solution, the equation can be easily solved as

$$0 = t^3 - 2t + 1 = (t-1)(t^2+t-1) = (t-1)(t - \frac{-1+\sqrt{5}}{2})(t - \frac{-1-\sqrt{5}}{2})$$

and so $\rho = \frac{-1+\sqrt{5}}{2} \in (0,1)$.

Queuing chain

Similar to branching chain, its property depends on the distribution f of arrivals

- (HW3 Q37) a queuing chain is irreducible iff $f(0) > 0$ and $f(0) + f(1) < 1$
- the recurrence probability $\rho = \rho_{00}$ for an irreducible chain solves $\rho = \Phi(\rho)$ with $\Phi(t) = \sum f(k)t^k$ being the generating function of the arrival distribution
- $\rho_{x0} = \rho^x$ on $x \geq 1$
- If irreducible, then the chain is recurrent iff $\mu = E_0(X_1) = \Phi'(1) \leq 1$

We will show the irreducible condition of queuing chain in HW3 Q37.

HW3 Q37

Suppose the arrival distribution f satisfies $f(0) > 0$ and $f(0) + f(1) < 1$. Then for some $k \geq 2$, $f(k) > 0$. So

- $\rho_{00} \geq P(0,0) = f(0) > 0$
- for $x > y \geq 0$, $\rho_{xy} \geq P(x, x-1)P(x-1, x-2)\dots P(y+1, y) = f(0)^{x-y} > 0$
- for $x > 0$, with $n > \frac{x-1}{k-1}$ sufficiently large such that $k + (n-1)(k-1) > x$,

$$\begin{aligned}\rho_{0x} &\geq \rho_{0,k+(n-1)(k-1)}\rho_{k+(n-1)(k-1),x} \\ &\geq P(0,k)P(k, k+(k-1))\dots P(k+(n-2)(k-1), k+(n-1)(k-1))\rho_{k+(n-1)(k-1),x} \\ &= f(k)^n \rho_{k+(n-1)(k-1),x} \\ &> 0\end{aligned}$$

so for all $x \geq 0$, x leads to 0 and 0 leads to x. This implies that x leads to y for all $x, y \geq 0$, and thus the chain is irreducible.

Suppose now that $f(0) = 0$ or $f(0) + f(1) = 1$.

- If $f(0) = 0$, then for all $x > 0$, $P(x, x-1) = 0$. By the structure of the chain, this implies that the chain random variables X_n must be non-decreasing $X_{n+1} \geq X_n$, and so $\rho_{x0} = 0$ for $x > y$.
- If $f(0) + f(1) = 1$, then for all $y > x > 0$, $P(x,y) = f(y-x+1) = 0$. By the structure of the chain, this implies that $\rho_{xy} = 0$ for all $y > x > 1$.

In both cases, the chain is reducible.