MATH4240 Tutorial 4

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Random walk

Consider yourself walking on a d-dimensional grid $\mathbb{Z}^d$ starting at the origin, and at each junction you pick a direction along the grid uniformly random and walk one step forward in that direction. Can you return to where you start?

Let X_n denote the location after n steps. It is easy to see that $X_{n+1} = X_n + \xi_n$ where $\xi_1, \ldots$ is iid random variables distributed uniformly random on all directions with $P(\xi_n = e_i) = P(\xi_n = -e_i) = \frac{1}{2d}$ for all n, i. In particular, $\{X_n\}$ is a Markov chain.

1D random walk, with one-step argument

For $d = 1$, the transition function is $P(x, x-1) = P(x, x+1) = \frac{1}{2}$ for each $x \in \mathbb{Z}$.

One-step argument on the recurrent probability ρ_{x0} gives

$$\begin{aligned}
\rho_{x0} &= P(x, x-1)\rho_{x-1,0} + P(x, x+1)\rho_{x+1,0} \\
&= \frac{1}{2}\rho_{x-1,0} + \frac{1}{2}\rho_{x+1,0}, \quad x \in \mathbb{Z} \setminus \{1, -1\} \\
\rho_{10} &= P(1, 0) + P(1, 2)\rho_{20} = \frac{1}{2} + \frac{1}{2}\rho_{20} \\
\rho_{-1,0} &= P(-1, 0) + P(-1, -2)\rho_{-2,0} = \frac{1}{2} + \frac{1}{2}\rho_{-2,0}
\end{aligned}$$

Rearranging the first equation gives

$$\begin{aligned}
\rho_{x+1,0} - \rho_{x0} &= \rho_{x0} - \rho_{x-1,0}, \quad x \notin \{1, -1\} \\
\text{so} \quad \rho_{x0} &= \rho_{10} + (x-1)(\rho_{20} - \rho_{10}), \quad x \geq 1 \\
\rho_{x0} &= \rho_{-1,0} - (x+1)(\rho_{-2,0} - \rho_{-1,0}), \quad x \leq -1
\end{aligned}$$

Since $\rho_{x0} \in [0, 1]$ for all x, we must have $\rho_{20} - \rho_{10} = \rho_{-2,0} - \rho_{-1,0} = 0$ and so $\rho_{x0} = \rho_{10}$ for $x \geq 1$ and $\rho_{x0} = \rho_{-1,0}$ for $x \leq -1$. Rearranging the equations for $\rho_{10}, \rho_{-1,0}$ gives

$$\begin{aligned}
\rho_{10} - \rho_{20} &= 1 - \rho_{10} \\
\text{so} \quad \rho_{10} &= 1 \\
\rho_{-1,0} - \rho_{-2,0} &= 1 - \rho_{-1,0} \\
\text{so} \quad \rho_{-1,0} &= 1
\end{aligned}$$

Using the equation for ρ_{00} then implies

$$\rho_{00} = \frac{1}{2}\rho_{-1,0} + \frac{1}{2}\rho_{10} = 1$$

In particular, 0 is a recurrent state. As the chain is irreducible, every state is recurrent, and $\rho_{xy} = 1$ for all x, y.

Unfortunately, this approach is not really well-suited for higher dimension cases as the recurrence relation becomes messy to handle.

1D random walk, with Stirling

Alternatively, we can consider the expected number of visits $E_0(N(0))$, which is infinite iff 0 is a recurrent state.

You can only return to the origin if you take even steps and takes the same amount of left steps and right steps. Hence

$$P^n(0,0) = \begin{cases} 0 & \text{if } n = 2k+1,\ k \in \mathbb{N} \\ \binom{2k}{k} P\,(\text{left})^k\, P\,(\text{right})^k & \text{if } n = 2k,\ k \in \mathbb{N} \end{cases} = \begin{cases} 0 & \text{if } n = 2k+1,\ k \in \mathbb{N} \\ 2^{-2k}\frac{(2k)!}{(k!)^2} & \text{if } n = 2k,\ k \in \mathbb{N} \end{cases}$$

And so the expected number of visits $E_0(N(0)) = \sum_{n=1}^{\infty} P^n(0,0) = \sum_{k=1}^{\infty} P^{2k}(0,0) = \sum_{k=1}^{\infty} 2^{-2k}\frac{(2k)!}{(k!)^2}$. Using Stirling's approximation

$$n! \approx C(n/e)^n\sqrt{n} \quad \text{for some constant } C > 0$$

we have

$$2^{-2k}\frac{(2k)!}{(k!)^2} \approx 2^{-2k}\frac{C(2k/e)^{2k}\sqrt{2k}}{(C(k/e)^k\sqrt{k})^2} = C^{-1}\sqrt{2}k^{-1/2}$$

which implies that $E_0(T_0) = \sum_{n=1}^{\infty} 2^{-2k}\frac{(2k)!}{(k!)^2} \approx C^{-1}\sqrt{2}\sum_{k=1}^{\infty} k^{-1/2} = \infty$. This implies that 0 is a recurrent state, and so $\rho_{00} = 1$.

Higher dimensional random walk, with Stirling

For $d = 2$, similar to $d = 1$ we can only return to the origin with even steps, and furthermore we must have k steps east, k steps west, $n-k$ steps north, $n-k$ steps south among these $2n$ steps for some $k \in \{0, 1, \ldots, n\}$. So counting paths gives

$$\begin{aligned} P^{2k}(0,0) &= (1/4)^{2k}\sum_{k=0}^{n}\frac{(2n)!}{k!k!(n-k)!(n-k)!} \\ &= 2^{-4k}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2 \\ &= 2^{-4k}\binom{2n}{n}^2 \approx 2C^{-1}k^{-1} \end{aligned}$$

So $E_0(N(0)) = \sum_{n=1}^{\infty} P^n(0,0) \approx 2C^{-1}\sum_{k=1}^{\infty} k^{-1} = \infty$, which implies that 0 is recurrent.

Alternatively, we can also consider the component random variables $X_n = (X_{n,1}, X_{n,2})$ with respective steps $\xi_n = (\xi_{n,1}, \xi_{n,2})$. However, $\xi_{1,n}, \xi_{2,n}$ as random variables on $\{-1, 0, 1\}$ are dependent (when $\xi_{n,1} \neq 0$, we must have $\xi_{n,2} = 0$), so it will be more convenient to work on

$$Y_{n,1} = X_{n,1} + X_{n,2}, \quad Y_{n,2} = X_{n,1} - X_{n,2}$$

Their increments $\zeta_{n,i} = Y_{n+1,i} - Y_{n,i}$ are then random variables on $\{-1, 1\}$, and $\zeta_{n,1}, \zeta_{n,2}$ are independent (try to prove it!). This means that $\{Y_{n,1}\}, \{Y_{n,2}\}$ are two independent 1D random walk. This gives the same result as counting, and also explains why $P^n_{d=2}(0,0) = (P^n_{d=1}(0,0))^2$.

Similarly, for $d = 3$, $P^{2n+1}(0,0) = 0$ and

$$\begin{aligned} P^{2n}(0,0) &= (\frac{1}{2\cdot 3})^{2n}\sum_{i+j\leq n}\frac{(2n)!}{(i!j!(n-i-j)!)^2} \\ &= (1/6)^{2n}\binom{2n}{n}\sum_{i+j+k=n}(\frac{n!}{i!j!k!})^2 \end{aligned}$$

Note that if $i > j+1$, then

$$(i-1)!(j+1)!k! = i!j!k!\frac{j+1}{i} < i!j!k!$$

with $|\ (i-1)-(j+1)\ | = i-j-2 < i-j = |\ i-j\ |$. This means that $i!j!k!$ is minimized when the indices are closet, which gives $i!j!k! \geq (\lfloor n/3 \rfloor !)^3$ and so

$$\begin{aligned}
P^{2n}(0,0) &\leq (1/2)^{2n}\binom{2n}{n}\frac{n!}{(\lfloor n/3 \rfloor !)^3}3^{-n}\sum_{i+j+k=n}3^{-n}\frac{n!}{i!j!k!} \\
&= (1/2)^{2n}\binom{2n}{n}\frac{n!}{(\lfloor n/3 \rfloor !)^3}3^{-n}(3^{-1}+3^{-1}+3^{-1})^n \\
&\approx C'n^{-3/2}
\end{aligned}$$

This implies that

$$\sum_{n=1}^{\infty} P^{2n}(0,0) \lessapprox C'\sum_{n=1}^{\infty} n^{-3/2} < \infty$$

So 0 is a transient state.

With the same approach one can show that on $\mathbb{Z}^d$,

$$P^{2n}(0,0) \approx C_d n^{-d/2}$$

and so the chain is transient on $d \geq 3$.

Extra: Fourier approach

We can consider a more general walk. Assume that $P(x, x+k) = p(k)$ for each x, k, and consider $p_n(k) = P^n(0,k)$. By one-step argument and symmetry,

$$\begin{aligned}
p_1(k) &= p(k) \\
p_{n+1}(k) &= P^{n+1}(0,k) = \sum_{x\in\mathbb{Z}} P(0,x)P^n(x,k) = \sum_{x\in\mathbb{Z}} p(x)p_n(k-x)
\end{aligned}$$

In particular, if $\hat{p}_k(\theta) = \sum_{x\in\mathbb{Z}} p_k(x)e^{-ix\theta}$ and $\hat{p}(\theta) = \sum_{x\in\mathbb{Z}} p(x)e^{-ix\theta}$ are the Fourier transform, then

$$\begin{aligned}
\hat{p}_1(\theta) &= \sum_{x\in\mathbb{Z}} p_1(x)e^{-ix\theta} = \sum_{x\in\mathbb{Z}} p(x)e^{-ix\theta} = \hat{p}(\theta) \\
\hat{p}_{n+1}(\theta) &= \sum_{x\in\mathbb{Z}} p_{n+1}(x)e^{-ix\theta} = \sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}}(p(y)e^{-iy\theta})(p_n(x-y)e^{-i(x-y)\theta}) = \hat{p}(\theta)\hat{p}_n(\theta) \\
\text{and so}\quad \hat{p}_n(\theta) &= \hat{p}(\theta)^n
\end{aligned}$$

By the property of Fourier transform, $P^n(0,k) = p_n(k) = (2\pi)^{-1}\int_{-\pi}^{\pi}\hat{p}_n(\theta)e^{ik\theta}\ \mathrm{d}\theta = (2\pi)^{-1}\int_{-\pi}^{\pi}\hat{p}(\theta)^n e^{ik\theta}\ \mathrm{d}\theta$. In particular,

$$P^n(0,0) = (2\pi)^{-1}\int_{-\pi}^{\pi}\hat{p}(\theta)^n\ \mathrm{d}\theta$$

In the case $P(x,x-1) = P(x,x+1) = \frac{1}{2}$, $\hat{p}(\theta) = \frac{1}{2}e^{-i\theta} + \frac{1}{2}e^{i\theta} = \cos\theta$, and you can show that $P^{2n}(0,0) \geq C_1 n^{-1/2}$ on n sufficiently large by cutting off the peaks and e.g. Laplace method.

This approach is able to handle high dimensional cases and even more general random walk where the step is not a discrete random variable (e.g. Gaussian distributed step), but estimating the integral (usually) becomes complicated and often requires advanced knowledge in Fourier analysis.

Within the scope of this course, there is no need to use such approach.

Transient part of transition matrix

Consider a (finite) Markov chain on the state space $\mathscr{S}$. We know that $\mathscr{S}$ decomposes as

$$\mathscr{S} = C_1 \cup C_2 \cup \ldots \cup C_k \cup \mathscr{S}_T$$

where each C_i is irreducible closed and $\mathscr{S}_T$ is the set of transient states. The transition matrix in this decomposition is then (see lecture notes)

$$P = \begin{array}{c} \\ C_1 \\ C_2 \\ \vdots \\ C_k \\ \mathscr{S}_T \end{array} \begin{array}{c} \begin{array}{ccccc} C_1 & C_2 & \dots & C_k & \mathscr{S}_T \end{array} \\ \begin{bmatrix} P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_k & 0 \\ Q_1 & Q_2 & \dots & Q_k & Q \end{bmatrix} \end{array}$$

where Q is the part of the transition matrix that transfer states insider $\mathscr{S}_T$.

Take two transient states $x, y \in \mathscr{S}_T$. Then

$$P^2(x,y) = \sum_{z \in \mathscr{S}} P(x,z)P(z,y)$$

Noting that $P(z,y) = 0$ if $z \in \bigcup C_i = \mathscr{S} \setminus \mathscr{S}_T$, we can simplify the relation as

$$P^2(x,y) = \sum_{z \in \mathscr{S}_T} P(x,z)P(z,y) = Q(x,z)Q(y,z) = Q^2(x,y)$$

Induction gives you $P^n(x,y) = Q^n(x,y)$.

Since x is transient, the expected number of visits $E_x(N(y)) = \sum_{n=1}^{\infty} P^n(x,y) = \frac{\rho_{xy}}{1-\rho_{yy}}$ is finite. This implies that $Q^n(x,y) = P^n(x,y) \xrightarrow{n\to\infty} 0$.

If Q has an eigenvalue λ with modulus $\mid \lambda \mid \geq 1$, then with a corresponding unit eigenvector v we have $\lambda^n v = Q^n v \xrightarrow{n\to\infty} 0$, from which contradiction arises.

This implies that every eigenvalue of Q must have modulus less than 1 (see lecture note).

Limit transition matrix on chains with only absorbing recurrent states

Consider a (finite) Markov chain where every recurrent states is an absorbing state. In this case the transition matrix is

$$P = \begin{array}{r} \\ \text{absorbing} \\ \mathscr{S}_T \end{array} \begin{array}{c} \begin{array}{cc} \text{absorbing} & \mathscr{S}_T \end{array} \\ \begin{bmatrix} I & 0 \\ S & Q \end{bmatrix} \end{array}$$

We already know that $Q^n \xrightarrow{n\to\infty} 0$, so what is P^n?

By direct computation,

$$\begin{aligned} P^2 &= \begin{bmatrix} I & 0 \\ S & Q \end{bmatrix} \begin{bmatrix} I & 0 \\ S & Q \end{bmatrix} = \begin{bmatrix} I & 0 \\ S + QS & Q^2 \end{bmatrix} \\ P^3 &= PP^2 = \begin{bmatrix} I & 0 \\ S + QS + Q^2 S & Q^3 \end{bmatrix} \\ \text{by induction} \quad P^n &= \begin{bmatrix} I & 0 \\ (\sum_{k=0}^{n} Q^k)S & Q^n \end{bmatrix} \end{aligned}$$

so passing the limit gives

$$\lim_{n\to\infty} P^n = \begin{bmatrix} I & 0 \\ (\sum_{k=0}^{\infty} Q^k)S & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ (I-Q)^{-1}S & 0 \end{bmatrix}$$

where $I - Q$ is invertible as all eigenvalues of Q have moduli less than 1.