MATH4240 Tutorial 3

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One-step argument

(HW2 Optional Part Q9) Consider a *finite* Markov chain X_n, $n \geq 0$. Let us use the formula 29

$$P_x(T_y = n+1) = \sum_{z \neq y} P(x,z) P_z(T_y = n) \quad \text{for } n \geq 1$$

from textbook to verify the following identities:

$$P_x(T_y \leq n+1) = P(x,y) + \sum_{x \neq y} P(x,z) P_z(T_y \leq n) \quad \text{for } n \geq 0$$

$$\rho_{xy} = P(x,y) + \sum_{z \neq y} P(x,z) \rho_{zy}$$

Since the chain has only finitely many states,

$$\begin{aligned} P_x(T_y \leq n+1) &= P_x(T_y = 1) + \sum_{k=1}^{n} P_x(T_y = k+1) \\ &= P(x,y) + \sum_{k=1}^{n} \sum_{z \neq y} P(x,z) P_z(T_y = k) \\ &= P(x,y) + \sum_{z \neq y} P(x,z) \sum_{k=1}^{n} P_z(T_y = k) \\ &= P(x,y) + \sum_{z \neq y} P_z(T_y \leq n) \end{aligned}$$

and similarly

$$\begin{aligned} \rho_{xy} = P_x(T_y < \infty) &= \lim_{n \to \infty} P_x(T_y \leq n+1) \\ &= P(x,y) + \lim_{n \to \infty} \sum_{z \neq y} P(x,z) P_z(T_y \leq n) \\ &= P(x,y) + \sum_{z \neq y} P(x,z) \lim_{n \to \infty} P_z(T_y \leq n) \\ &= P(x,y) + \sum_{z \neq y} P(x,z) \rho_{z,y} \end{aligned}$$

We can derive a similar formula for expected hitting time:

$$E_x(T_y) = \rho_{xy} + \sum_{z \neq y} P(x,z) E_z(T_y)$$

by the same approach

$$
\begin{aligned}
E_x(T_y) &= \sum_{n=0}^{\infty}(n+1)P_x(T_y = n+1) \\
&= P_x(T_y = 1) + \sum_{n=1}^{\infty}(n+1)\sum_{z\neq y} P(x,z)P_z(T_y = n) \\
&= P(x,y) + \sum_{z\neq y} P(x,z)\sum_{n=1}^{\infty}(n+1)P_z(T_y = n) \\
&= P(x,y) + \sum_{z\neq y} P(x,z)\left(\sum_{n=1}^{\infty} nP_z(T_y = n) + \sum_{n=1}^{\infty} P_z(T_y = n)\right) \\
&= P(x,y) + \sum_{z\neq y} P(x,z)(E_z(T_y) + \rho_{zy}) \\
&= \rho_{xy} + \sum_{z\neq y} P(x,z)E_z(T_y)
\end{aligned}
$$

If the chain is irreducible, then $\rho_{xy} = 1$ for all x, y, so we have

$$
E_x(T_y) = 1 + \sum_{z\neq y} P(x,z)E_z(T_y)
$$

In fact, a more general formula holds (try to prove it yourself; compare with formula 44 in textbook)

$$
E_x(T_C) = \rho_C(x) + \sum_{z\notin C} P(x,z)E_z(T_C)
$$

Without the finite chain assumption, the proofs are still basically same, except more justification is needed to deal with the operations on infinite sums.

Example 1. Consider the Ehrenfest chain on $d = 3$, state space $\mathscr{S} = \{0,1,2,3\}$ and transition matrix

$$
P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

Let us compute ρ_{x0} and $E_x(T_0)$ for each $x \in \mathscr{S}$.

Note that the chain is irreducible. By the formulae,

$$
\begin{aligned}
\rho_{00} &= P(0,0) + P(0,1)\rho_{10} \\
\rho_{10} &= P(1,0) + P(1,2)\rho_{20} \\
\rho_{20} &= P(2,0) + P(2,1)\rho_{10} + P(2,3)\rho_{30} \\
\rho_{30} &= P(3,0) + P(3,2)\rho_{20}
\end{aligned}
$$

and

$$
\begin{aligned}
E_0(T_0) &= 1 + P(0,1)E_1(T_0) \\
E_1(T_0) &= 1 + P(1,2)E_2(T_0) \\
E_2(T_0) &= 1 + P(2,1)E_1(T_0) + P(2,3)E_3(T_0) \\
E_3(T_0) &= 1 + P(3,2)E_2(T_0)
\end{aligned}
$$

Note that there is no ρ_{00} and no $E_0(T_0)$ term on RHS.

We can write this as a linear system with matrices that can be solved with standard approaches (Gauss elimination, computing the inverse, etc.)

$$\begin{pmatrix}\rho_{00}\\ \rho_{10}\\ \rho_{20}\\ \rho_{30}\end{pmatrix} = \begin{pmatrix}0\\ 1/3\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 2/3 & 0\\ 0 & 2/3 & 0 & 1/3\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\rho_{00}\\ \rho_{10}\\ \rho_{20}\\ \rho_{30}\end{pmatrix}$$

$$\begin{pmatrix}\rho_{00}\\ \rho_{10}\\ \rho_{20}\\ \rho_{30}\end{pmatrix} = \left(I_4 - \begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 2/3 & 0\\ 0 & 2/3 & 0 & 1/3\\ 0 & 0 & 1 & 0\end{pmatrix}\right)^{-1}\begin{pmatrix}0\\ 1/3\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}1 & 3 & 3 & 1\\ 0 & 3 & 3 & 1\\ 0 & 3 & 9/2 & 3/2\\ 0 & 3 & 9/2 & 5/2\end{pmatrix}\begin{pmatrix}0\\ 1/3\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 1\\ 1\\ 1\end{pmatrix}$$

The result is unsurprising as the chain is finite and irreducible. Notice that the coefficient matrix is exactly the transition matrix with its first column (corresponding to state 0) zeroed out.

Similarly,

$$\begin{pmatrix}E_0(T_0)\\ E_1(T_0)\\ E_2(T_0)\\ E_3(T_0)\end{pmatrix} = \begin{pmatrix}\rho_{00}\\ \rho_{10}\\ \rho_{20}\\ \rho_{30}\end{pmatrix} + \begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 2/3 & 0\\ 0 & 2/3 & 0 & 1/3\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}E_0(T_0)\\ E_1(T_0)\\ E_2(T_0)\\ E_3(T_0)\end{pmatrix}$$

$$\begin{pmatrix}E_0(T_0)\\ E_1(T_0)\\ E_2(T_0)\\ E_3(T_0)\end{pmatrix} = \left(I_4 - \begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 2/3 & 0\\ 0 & 2/3 & 0 & 1/3\\ 0 & 0 & 1 & 0\end{pmatrix}\right)^{-1}\begin{pmatrix}1\\ 1\\ 1\\ 1\end{pmatrix} = \begin{pmatrix}1 & 3 & 3 & 1\\ 0 & 3 & 3 & 1\\ 0 & 3 & 9/2 & 3/2\\ 0 & 3 & 9/2 & 5/2\end{pmatrix}\begin{pmatrix}1\\ 1\\ 1\\ 1\end{pmatrix} = \begin{pmatrix}8\\ 7\\ 9\\ 10\end{pmatrix}$$

Notice that $E_0(T_0) = E_1(T_0) + 1$ and $E_3(T_0) = E_2(T_0) + 1$, which is obvious from the structure of the chain.

Probability of ruining and expected duration in a fair gambler's ruin

Consider the gambler's ruin chain with fair probability

$$P(x, x+1) = P(x, x-1) = 1/2, \quad x \in \{1, 2, \ldots, N-1\}$$
$$P(0,0) = P(N,N) = 1$$

Let us find the ruin probability ρ_{x0} and the expected duration $E_x(T)$ for each initial state $x \in \{1, 2, \ldots, N-1\}$. Here $T = \min(T_0, T_N)$ is the time the chain enters an absorbing state (0 or N).

Using the formula noting that $\rho_{00} = 1$,

$$\begin{aligned}\rho_{x0} &= P(x,0) + P(x,x-1)\rho_{x-1,0} + P(x,x+1)\rho_{x+1,0}\\ &= \frac{1}{2}\rho_{x-1,0} + \frac{1}{2}\rho_{x+1,0}\end{aligned}$$

for $x \in \{1, \ldots, N-1\}$. Rearranging the equations with $h(x) = \rho_{x0} - \rho_{x-1,0}$,

$$\rho_{x+1,0} - \rho_{x0} = \rho_{x0} - \rho_{x-1,0}$$
$$h(x+1) = h(x)$$

Summing h from 1 to x gives

$$\rho_{x0} - 1 = \rho_{x0} - \rho_{00} = \sum_{i=1}^{x} h(i) = xh(1) = x(\rho_{10} - 1)$$

Noting that $\rho_{N0} = 0$, we have $-1 = \rho_{N0} - \rho_{00} = N(\rho_{10} - 1)$ and so $\rho_{10} = 1 - \frac{1}{N}$,

$$\rho_{x0} = 1 - \frac{x}{N}$$

A similar process gives you $\rho_{xN} = x/N$.

We can use the same approach to compute $E_x(T)$ for each $x \in \{1, 2, \dots, N-1\}$.

For the sake of consistency, we may write $E_0(T) = E_N(T) = 0$. Then since the chain is irreducible,

$$\begin{aligned} E_x(T) &= 1 + P(x, x-1)E_{x-1}(T) + P(x, x+1)E_{x+1}(T) \\ &= 1 + \frac{1}{2}E_{x-1}(T) + \frac{1}{2}E_{x+1}(T) \end{aligned}$$

for $x \in \{1, 2, \dots, N-1\}$.

Rearranging the equation, on $h(x) = E_x(T) - E_{x-1}(T)$ we have

$$\begin{aligned} E_{x+1}(T) - E_x(T) &= E_x(T) - E_{x-1}(T) - 2 \\ h(x+1) &= h(x) - 2 \\ h(x) &= h(1) - 2(x-1) \end{aligned}$$

Summing h from 1 to x we have

$$\begin{aligned} E_x(T) = E_x(T) - E_0(T) &= \sum_{i=1}^{x} h(i) \\ &= \sum_{i=1}^{x} (h(1) - 2(i-1)) \\ &= xh(1) - x(x-1) \end{aligned}$$

Noting that $E_N(T) = 0$, we have $0 = Nh(1) - N(N-1)$ and so $h(1) = N - 1$,

$$E_x(T) = x(N-1) - x(x-1) = x(N-x)$$

If the probability is not fair ($\frac{1}{2}/\frac{1}{2}$), the equations can still be handled with a similar method (to be covered in later lectures).