MATH4240 Midterm 1 Reference Solution

1. (10 points) Let $\{X_n\}_{n\geq 0}$ be a Markov chain with state space $\{1,2,3\}$ and transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{2}{5} & 0 & \frac{3}{5} \end{pmatrix}$$

and initial distribution $\pi = (\frac{2}{5}, \frac{1}{5}, \frac{2}{5})$. Compute the following probability:

$$P(X_1 = 2,\ X_2 = 2,\ X_3 = 2,\ X_4 = 1,\ X_5 = 3).$$

Solution:

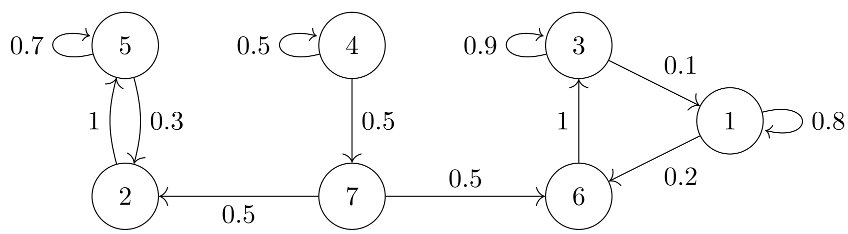
$$\begin{aligned}
&P(X_1 = 2,\ X_2 = 2,\ X_3 = 2,\ X_4 = 1,\ X_5 = 3) \\
&= P(X_1 = 2)P(X_2 = 2 \mid X_1 = 2)P(X_3 = 2 \mid X_2 = 2)P(X_4 = 1 \mid X_3 = 2)P(X_5 = 3 \mid X_4 = 1) \\
&= (P(X_0 = 1)P(X_1 = 2 \mid X_0 = 1) + P(X_0 = 2)P(X_1 = 2 \mid X_0 = 2) + P(X_0 = 3)P(X_1 = 2 \mid X_0 = 3)) \\
&\qquad \times P_{22}P_{22}P_{21}P_{13} \\
&= (\pi_1 P_{12} + \pi_2 P_{22} + \pi_3 P_{32})P_{22}P_{22}P_{21}P_{13} \\
&= (\frac{2}{5} \times \frac{1}{3} + \frac{1}{5} \times \frac{3}{4} + \frac{2}{5} \times 0) \times \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} \times \frac{2}{3} = \frac{17}{640} \approx 0.0266
\end{aligned}$$

2. (20 points) Consider a Markov chain with the following transition matrix:

$$P = \begin{pmatrix} 0.8 & 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0.1 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0..5 \\ 0 & 0.3 & 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 \end{pmatrix}$$

Name the state space as $\{1, 2, \cdots, 7\}$, draw the transition graph and classify the states by decomposing the state space to be the disjoint union of transient sets and closed irreducible sets of recurrent states. The detailed reasonings as given in lectures are necessary.

Solution: The transition graph is



We can see that $C_1 = \{1, 3, 6\}$ and $C_2 = \{2, 5\}$ are irreducible and closed:

- Since every state in C_1 only communicates to states inside C_1, C_1 is closed. Also, there is a circuit $1 \to 6 \to 3 \to 1$ that traverses every state in C_1, so C_1 is irreducible.
- Since every state in C_2 only communicates to states inside C_2, C_2 is closed. Also, there is a circuit $2 \to 5 \to 2$ that traverses every state in C_2, so C_2 is irreducible.

As C_1, C_2 are finite sets, they are closed irreducible sets of recurrent states.

Since $S_T = S \setminus (C_1 \cup C_2) = \{4, 7\}$ is disjoint from C_1, C_2 and there is a path $4 \to 7 \to 2$ that traverses all states in S_T and ends at a state in an irreducible closed set C_2, $4, 7$ are transient states.

Therefore, $S = C_1 \cup C_2 \cup S_T$ is a disjoint union where $C_1 = \{1, 3, 6\}, C_2 = \{2, 5\}$ are closed irreducible sets of recurrent states and $S_T = \{4, 7\}$ is set of transient states.

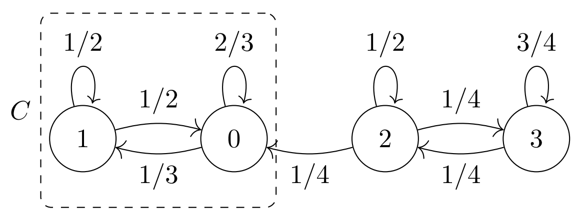
3. (20 points) Let $\{X_n\}_{n \geq 9}$ be a Markov chain with the state space $S = \{0, 1, 2, 3\}$ and transition matrix

$$P = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix} \end{array}$$

(a) Find all probabilities $\rho_{xy} = P_x(T_y < \infty)$ and white them in the matrix form $[\rho_{xy}]$.

(b) Find all values $E_x(N(y))$, the expected number of visits to y from x, and write them in the matrix form $[E_x(N(y))]$.

Solution: We first draw the transition diagram:



It is easy to see that $C = \{0, 1\}$ is an irreducible closed set of recurrent states and $S_T = \{2, 3\}$ are the set of transient states.

(a) Since C is irreducible closed set of recurrent states, we have $\rho_{xy} = 1$ for $x, y \in C$ and $\rho_{xy} = 0$ for $x \in C$ and $y \in S_T$. So

$$[\rho_{xy}] = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \end{array}$$

where $*$ are the entries that are not yet computed. For these entries, with one-step argument we have

$$\rho_{20} = P(2,0) + P(2,2)\rho_{20} + P(2,3)\rho_{30} = \frac{1}{4} + \frac{1}{2}\rho_{20} + \frac{1}{4}\rho_{30}$$
$$\rho_{30} = P(3,2)\rho_{20} + P(3,3)\rho_{30} = \frac{1}{4}\rho_{20} + \frac{3}{4}\rho_{30}$$

$$\rho_{21} = P(2,0)\rho_{01} + P(2,2)\rho_{21} + P(2,3)\rho_{31} = \frac{1}{4} + \frac{1}{2}\rho_{21} + \frac{1}{4}\rho_{31}$$
$$\rho_{31} = P(3,2)\rho_{21} + P(3,3)\rho_{31} = \frac{1}{4}\rho_{21} + \frac{3}{4}\rho_{31}$$

$$\rho_{22} = P(2,2) + P(2,3)\rho_{32} + P(2,0)\rho_{02} = \frac{1}{2} + \frac{1}{4}\rho_{32}$$
$$\rho_{32} = P(3,2) + P(3,3)\rho_{32} = \frac{1}{4} + \frac{3}{4}\rho_{32}$$

$$\rho_{23} = P(2,3) + P(2,2)\rho_{23} + P(2,0)\rho_{03} = \frac{1}{4} + \frac{1}{2}\rho_{23}$$
$$\rho_{33} = P(3,3) + P(3,2)\rho_{23} = \frac{3}{4} + \frac{1}{4}\rho_{23}$$

Solving these systems gives $\rho_{20} = \rho_{30} = \rho_{21} = \rho_{31} = \rho_{32} = 1$, $\rho_{22} = \frac{3}{4}$, $\rho_{23} = \frac{1}{2}$, $\rho_{33} = \frac{7}{8}$. Hence

$$[\rho_{xy}] = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 3/4 & 1/2 \\ 1 & 1 & 1 & 7/8 \end{bmatrix} \end{array}$$

(b) From lecture notes we have

$$E_x(N(y)) = \begin{cases} 0 & \text{if } y \text{ is recurrent and } \rho_{xy} = 0 \\ \infty & \text{if } y \text{ is recurrent and } \rho_{xy} \neq 0 \\ \frac{\rho_{xy}}{1-\rho_{yy}} & \text{if } y \text{ transient} \end{cases}$$

Since C is irreducible closed set of recurrent states, we have $\rho_{xy} = \infty$ for $x, y \in C = \{0, 1\}$ and $\rho_{xy} = 0$ for $x \in C$ and $y \in S_T$. So with the result from part (a),

$$[E_x(T_y)] = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \infty & \infty & \frac{0}{1-3/4} & \frac{0}{1-7/8} \\ \infty & \infty & \frac{0}{1-3/4} & \frac{0}{1-7/8} \\ \infty & \infty & \frac{3/4}{1-3/4} & \frac{1/2}{1-7/8} \\ \infty & \infty & \frac{1}{1-3/4} & \frac{7/8}{1-7/8} \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \infty & \infty & 0 & 0 \\ \infty & \infty & 0 & 0 \\ \infty & \infty & 3 & 4 \\ \infty & \infty & 4 & 7 \end{bmatrix} \end{array}$$

4. (10 points) Let X, Y be two independent random variables having Poisson distributions with rates λ_1, λ_2 respectively. Show that $Z \coloneqq X + Y$ has Poisson distribution with rate $\lambda_1 + \lambda_2$.

Solution: By assumption, X, Y have pdf $p_X(k) = e^{-\lambda_1}\frac{\lambda_1^k}{k!}$ and $p_Y(k) = e^{-\lambda_2}\frac{\lambda_2^k}{k!}$ on $k \in \mathbb{N}$ respectively. So the pdf of $Z = X + Y$ is

$$\begin{aligned} p_Z(k) = (p_X * p_Y)(k) &= \sum_{m=0}^{k} p_X(m) p_Y(k-m) \\ &= \sum_{m=0}^{k} e^{-\lambda_1}\frac{\lambda_1^m}{m!} e^{-\lambda_2}\frac{\lambda_2^{k-m}}{(k-m)!} \\ &= e^{-(\lambda_1+\lambda_2)}\frac{1}{k!}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}\lambda_1^m\lambda_2^{k-m} = e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^k}{k!} \end{aligned}$$

on $k \in \mathbb{N}$, so Z is of Poisson distribution with rate $\lambda_1 + \lambda_2$.

5. (15 points) When a basketball player makes a shot then he tries a harder shot the next time and hits (H) with probability 0.4, misses (M) with probability 0.6. When he misses he is more conservative the next time and hits (H) with probability 0.7, misses (M) with probability 0.3.

(a) Write the transition probability for the two states Markov chain with the state space $\{H, M\}$.

(b) Find the long-run probability he hits a shot.

Solution:

(a) By the question we have

$$P(H \mid H) = 0.4$$
$$P(M \mid H) = 0.6$$
$$P(H \mid M) = 0.7$$
$$P(M \mid M) = 0.3$$

which gives the transition matrix as

$$P = \begin{array}{c c} & \begin{array}{cc} H & M \end{array} \\ \begin{array}{c} H \\ M \end{array} & \begin{bmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{bmatrix} \end{array}$$

(b) To find $\lim_{n\to\infty} P^n$, we can diagonalize P. By direct computation, $P = QDQ^{-1}$ where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -3/10 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & -6 \\ 1 & 7 \end{pmatrix} \quad \text{with} \quad Q^{-1} = \frac{1}{13}\begin{pmatrix} 7 & 6 \\ -1 & 1 \end{pmatrix}$$

so

$$\lim_{n\to\infty} P^n = Q(\lim_{n\to\infty} D^n)Q^{-1} = Q\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}Q^{-1} = \begin{pmatrix} 7/13 & 6/13 \\ 7/13 & 6/13 \end{pmatrix}$$

which implies that the long-run probability of hitting a shot is 7/13.

6. (25 points) Consider a branching chain $\{X_n\}_{n=0}^{\infty}$ with the offspring distribution $(p_k)_{k\geq 0}$. Let $X_0 = 1$.

(a) Assume $p_0 = 0.25$, $p_1 = 0.4$, $p_2 = 0.35$. What is the probability that the population becomes extinct in the second generation (i.e., $X_2 = 0$ but $X_1 \neq 0$)? What is the probability that the population becomes extinct eventually?

(b) Assume $p_k = p(1-p)^k$, $k \geq 0$, where $0 < p < 1$. Find the extinction probability ρ.

Solution:

(a) Since $X_0 = 1$, we must have $X_1 \leq 2$. Then the probability of extinction in the second generation is

$$\begin{aligned}
\rho_2 &= P(X_2 = 0,\, X_1 \neq 0) \\
&= P(X_1 = 1,\, X_2 = 0) + P(X_1 = 2,\, X_2 = 0) \\
&= P(X_2 = 0 \mid X_1 = 1)P(X_1 = 1) + P(X_2 = 0 \mid X_1 = 2)P(X_1 = 2) \\
&= p_0 p_1 + p_0^2 p_2 \\
&= 0.25 \cdot 0.4 + 0.25^2 \cdot 0.35 \\
&= 39/320 \approx 0.12
\end{aligned}$$

The average number of offspring is

$$\begin{aligned}
\mu &= 0 \cdot p_0 + 1 \cdot p_1 + 2 \cdot p_2 \\
&= 0 \cdot 0.25 + 1 \cdot 0.4 + 2 \cdot 0.35 \\
&= 1.1 > 1
\end{aligned}$$

The generating function is

$$\Phi(t) = p_0 + p_1 t + p_2 t^2 = 0.25 + 0.4t + 0.35t^2$$

for which the fixed point equations is

$$\begin{aligned} \Phi(t) &= t \\ 0.05(t-1)(7t-5) &= 0 \end{aligned}$$

which means that the fixed points are $t = 1$ and $t = 5/7$.

This implies that the eventual extinction probability is $\rho = 5/7 \approx 0.71$.

(b) The generating function is

$$\Phi(t) = \sum_{k=0}^{\infty} p_k t^k = \sum_{k=0}^{\infty} p(1-p)^k t^k = \frac{p}{1-(1-p)t}$$

which gives the average number of offspring as

$$\mu = \Phi'(1) = \frac{p(1-p)}{(1-(1-p))^2} = \frac{1}{p} - 1$$

so if $p \geq 1/2$, $\mu = 1/p - 1 \leq 1$ and so the extinction probability is $\rho = 1$.

If $p < 1/2$, the fixed point equation is

$$\begin{aligned} \Phi(t) &= t \\ (t-1)((1-p)t-p) &= 0 \end{aligned}$$

and so the fixed points are $t = 1$, $t = \frac{p}{1-p} \in [0, 1]$. Hence the extinction probability is $\rho = \frac{p}{1-p}$.

Therefore the extinction probability is $\rho = \begin{cases} 1 & \text{if } p \geq 1/2 \\ \frac{p}{1-p} & \text{if } p < 1/2 \end{cases}$