MATH4240 Homework 3 Reference Solution

1 Compulsory Part

15. Let y be a transient state. Use (36) to show that for all x,

$$\sum_{n=0}^{\infty} P^n(x,y) \leq \sum_{n=0}^{\infty} P^n(y,y)$$

Solution: It suffices to consider the case $x \neq y$. Then

$$\sum_{n=0}^{\infty} P^n(x,y) = \sum_{n=1}^{\infty} P^n(x,y) = E_x(N(y)) = \frac{\rho_{xy}}{1-\rho_{yy}} \leq \frac{1}{1-\rho_{yy}} = 1 + \frac{\rho_{yy}}{1-\rho_{yy}} = 1 + E_y(N(y)) = \sum_{n=0}^{\infty} P^n(y,y)$$

18. Consider a Markov chain on the nonnegative integers such that, starting from x, the chain goes to state $x+1$ with probability p, $0 < p < 1$, and goes to state 0 with probability $1-p$.

(a) Show that this chain is irreducible

(b) Find $P_0(T_0 = n)$, $n \geq 1$

(c) Show that the chain is recurrent

Solution:

(a) Let x, y be two states. Then $\rho_{xy} \geq P(x,0)P(0,1)\ldots P(y-1,y) = (1-p)p^y > 0$. So x leads to y.
As x, y are arbitrary, the chain is irreducible.

(b) Starting from 0, there is only one path that goes to 0 only after n steps: $X_0 = 0, X_1 = 1, \ldots, X_{n-1} = n-1, X_n = 0$. So $P_0(T_0 = n) = P(0,1)P(1,2)\ldots P(n-2,n-1)P(n-1,0) = p^{n-1}(1-p)$

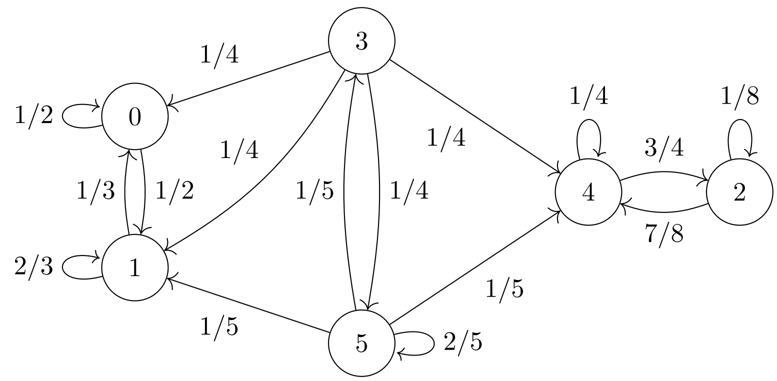
(c) Since the chain is irreducible, it suffices to show that 0 is a recurrent state.
By part (b), $\rho_{00} = \sum_{n=1}^{\infty} E_0(T_0 = n) = \sum_{n=1}^{\infty} p^{n-1}(1-p) = 1$. So 0 is a recurrent state. Therefore the chain is recurrent.

20(b). Consider the Markov chain on $\{0, 1, \ldots, 5\}$ having transition matrix

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix} \end{array}$$

Find $\rho_{\{0,1\}}(x)$, $x = 0, \ldots, 5$

Solution: As in HW2, the transition diagram is



and $\{0,1\}$, $\{2,4\}$ are irreducible closed sets.

Easy to see that $\rho_{\{0,1\}}(0) = \rho_{\{0,1\}}(1) = 1$, and $\rho_{\{0,1\}}(2) = \rho_{\{0,1\}}(4) = 0$.

By the one-step argument,

$$\rho_{\{0,1\}}(3) = P(3,0) + P(3,1) + P(3,5)\rho_{\{0,1\}}(5) = \frac{1}{2} + \frac{1}{4}\rho_{\{0,1\}}(5)$$
$$\rho_{\{0,1\}}(5) = P(5,1) + P(5,3)\rho_{\{0,1\}}(3) + P(5,5)\rho_{\{0,1\}}(5) = \frac{1}{5} + \frac{1}{5}\rho_{\{0,1\}}(3) + \frac{2}{5}\rho_{\{0,1\}}(5)$$

Solving this linear system gives $\rho_{\{0,1\}}(3) = \frac{7}{11}$ and $\rho_{\{0,1\}}(5) = \frac{6}{11}$.

Note

	0	1	2	3	4	5
$\rho_{\{0,1\}}(x)$	1	1	0	7/11	0	6/11

We can also reduce the chain as

$$\tilde{P} = \begin{array}{c} \\ \{0,1\} \\ \{2,4\} \\ 3 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} \{0,1\} & \{2,4\} & 3 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 1/4 & 0 & 1/4 \\ 1/5 & 1/5 & 1/5 & 2/5 \end{bmatrix} \end{array}$$

which gives

$$\begin{pmatrix} \rho_{\{0,1\}}(3) & \rho_{\{2,4\}}(3) \\ \rho_{\{0,1\}}(5) & \rho_{\{2,5\}}(5) \end{pmatrix} = \left(I - \begin{pmatrix} 0 & 1/4 \\ 1/5 & 2/5 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1/2 & 1/4 \\ 1/5 & 1/5 \end{pmatrix} = \begin{pmatrix} 7/11 & 4/11 \\ 6/11 & 5/11 \end{pmatrix}$$

24. Consider a gambler's ruin chain on $\{0, 1, \ldots, d\}$. Find

$$P_x(T_0 < T_d), \quad 0 < x < d$$

Solution: Recall that the transition function for $x \in \{1, \ldots, d-1\}$ is

$$P(x,y) = \begin{cases} p & y = x+1 \\ q & y = x-1 \\ 0 & \text{other} \end{cases}$$

with $q = 1 - p$.

As the chain is just a birth and death chain with constant $p_x = p$, $q_x = q$, applying the formula gives

$$P_x(T_0 < T_d) = \frac{\sum_{y=x}^{d-1} \gamma_y}{\sum_{y=0}^{d-1} \gamma_y}$$

with $\gamma_y = \frac{q_1 \cdots q_y}{p_1 \cdots p_y} = (\frac{q}{p})^y$.

- If $p = \frac{1}{2}$, we have $\gamma_y = (\frac{q}{p})^y = 1$, so $\sum_{y=x}^{d-1} \gamma_y = d - x$, $\sum_{y=0}^{d-1} \gamma_y = d$
- If $p \neq \frac{1}{2}$, we have $\gamma_y \neq 1$ and so $\sum_{y=x}^{d-1} \gamma_y = \frac{(q/p)^x - (q/p)^d}{1-q/p}$, $\sum_{y=0}^{d-1} \gamma_y = \frac{1-(q/p)^d}{1-q/p}$

This implies that

$$P_x(T_0 < T_d) = \begin{cases} 1 - \frac{x}{d} & \text{if } p = 1/2 \\ \frac{(q/p)^x - (q/p)^d}{1-(q/p)^d} & \text{if } p \neq 1/2 \end{cases}$$

26. Consider a birth and death chain on the nonnegative integers such that $p_x > 0$ and $q_x > 0$ for $x \geq 1$
    (a) Show that if $\sum_{y=0}^{\infty} \gamma_y = \infty$, then $\rho_{x0} = 1$, $x \geq 1$
    (b) Show that if $\sum_{y=0}^{\infty} \gamma_y < \infty$, then

$$\rho_{x0} = \frac{\sum_{y=x}^{\infty} \gamma_y}{\sum_{y=0}^{\infty} \gamma_y}, \quad x \geq 1$$

Solution: Since the chain has infinitely states, $T_0 < \infty$ iff there is some state $n > x$ such that $T_0 < T_n$ (where T_n can be infinite). This implies that $\{T_0 < \infty\} = \{T_0 < T_n \text{ for some } n > x\} = \bigcup_{n>x} \{T_0 < T_n\}$.

Furthermore, for $n > x$, you can only reach state $n+1$ after reaching state n, so $\{T_0 < T_n\}$ is increasing. This means that $\rho_{x0} = P_x(T_0 < \infty) = P_x(\bigcup_{n>x} \{T_0 < T_n\}) = \lim_{n\to\infty} P_x(T_0 < T_n) = 1 - (\sum_{y=0}^{x-1} \gamma_y) \lim_{n\to\infty} (\sum_{y=0}^{n-1} \gamma_y)^{-1}$.

(a) If $\sum_{y=0}^{\infty} \gamma_y = \infty$, then $\rho_{x0} = 1 - (\sum_{y=0}^{x-1} \gamma_y) \lim_{n\to\infty} (\sum_{y=0}^{n-1} \gamma_y)^{-1} = 1$ for all $x \geq 1$

(b) If $\sum_{y=0}^{\infty} \gamma_y < \infty$, then $\rho_{x0} = 1 - (\sum_{y=0}^{x-1} \gamma_y) \lim_{n\to\infty} (\sum_{y=0}^{n-1} \gamma_y)^{-1} = 1 - \frac{\sum_{y=0}^{x-1} \gamma_y}{\sum_{y=0}^{n-1} \gamma_y} = \frac{\sum_{y=x}^{\infty} \gamma_y}{\sum_{y=0}^{n-1} \gamma_y}$ for all $x \geq 1$

27. Consider a gambler's ruin chain on $\{0, 1, 2, \ldots\}$
    (a) Show that if $q \geq p$, then $\rho_{x0} = 1$, $x \geq 1$
    (b) Show that if $q < p$, then $\rho_{x0} = (q/p)^x$, $x \geq 1$

Solution: $\gamma_y = \prod_{x=1}^{y} \frac{q_x}{p_x} = (q/p)^y$.

(a) If $q \geq p$, $\gamma_y = (q/p)^y \geq 1$, so $\sum_{y=0}^{\infty} \gamma_y \geq \sum_{y=0}^{\infty} 1 = \infty$. By Q26(a), $\rho_{x0} = 1$ for each $x \geq 1$.

(b) If $q < p$, $\sum_{y=0}^{\infty} \gamma_y = \sum_{y=0}^{\infty} (q/p)^y = \frac{1}{1-q/p} < \infty$. So by Q26(b), $\rho_{x0} = \frac{\sum_{y=x}^{\infty} \gamma_y}{\sum_{y=0}^{\infty} \gamma_y} = \frac{(q/p)^x}{1-q/p} / \frac{1}{1-q/p} = (q/p)^x$ for each $x \geq 1$.

29. Consider an irreducible birth and death chain on the nonnegative integers such that

$$\frac{q_x}{p_x} = \left(\frac{x}{x+1}\right)^2, \quad x \geq 1$$

   (a) Show that this chain is transient
   (b) Find ρ_{x0}, $x \geq 1$

Solution:

(a) For each $y \geq 1$,

$$\gamma_y = \prod_{x=1}^{y} \frac{q_x}{p_x} = \left(\prod_{x=1}^{y} \frac{x}{x+1}\right)^2 = \frac{1}{(y+1)^2}$$

Then $\sum_{x=0}^{\infty} \gamma_x = 1 + \sum_{n=2}^{\infty} n^{-2} = \pi^2/6 < \infty$. So the chain is transient.

(b) By Q26(b), $\rho_{x0} = \frac{\sum_{y=x}^{\infty} \gamma_y}{\sum_{y=0}^{\infty} \gamma_y} = 1 - \frac{\sum_{y=0}^{x} \gamma_y}{\sum_{y=0}^{\infty} \gamma_y} = 1 - \frac{6}{\pi^2} \sum_{n=1}^{x+1} n^{-2}$ for each $x \geq 1$.

Note

There does not seem to be a (simple elementary) closed form expression for this sum.

32. Consider the branching chain described in Example 14. If a given man has two boys and one girl, what is the probability that his male line will continue forever?

Solution: Recall that in Example 14, every man has exactly 3 children, each independently has probability $p = 1/2$ of being a boy. From the textbook, the distinction probability is solved as $\rho = \sqrt{5} - 2$.

Therefore, $P\,(\text{continue forever}) = 1 - P\,(\text{both boys extinct}) = 1 - \rho^2 = 4(\sqrt{5} - 2) \approx 0.9442$

Note

For completeness, the extinction probability can be computed via the generating function of the offspring distribution: $\Phi(t) = \sum_{k=0}^{\infty} p_k t^k = \sum_{k=0}^{3} \binom{3}{k} 2^{-3} t^k = 2^{-3}(1+t)^3$. As $\Phi'(1) = 3/2 > 1$ and the solutions of $t = \Phi(t)$ are $t = 1$, $t = -\sqrt{5} - 2 < 0$ and $t = \sqrt{5} - 2 \in [0, 1]$, $\rho = \sqrt{5} - 2$.

34. Consider a branching chain with $f(x) = p(1-p)^x$, $x \geq 0$, where $0 < p < 1$. Show that $\rho = 1$ if $p \geq 1/2$ and that $\rho = p/(1-p)$ if $p < 1/2$

Solution: The generating function of f is $\Phi(t) = \sum_{n=0}^{\infty} f(n) t^n = \sum_{n=0}^{\infty} p(1-p)^n t^n = \frac{p}{1-(1-p)t}$ with $\mu = \Phi'(1) = \frac{1}{p} - 1$.

- If $p \geq 1/2$, $\mu = \frac{1}{p} - 1 \leq 1$, so $\rho = 1$
- If $p < 1/2$, $\mu = \frac{1}{p} - 1 = \frac{1-p}{p} > 1$. As the equation $t = \Phi(t) = \frac{p}{1-(1-p)t}$ then has solutions $t = 1$ and $t = \frac{p}{1-p} \in [0, 1]$, so $\rho = \frac{p}{1-p}$.

36(a). Let X_n, $n \geq 0$, be a branching chain and suppose that the associated random variable ξ has finite variance σ^2. Show that

$$E\left(X_{n+1}^2 \middle| X_n = x\right) = x\sigma^2 + x^2\mu^2$$

Solution: Let ξ_i be the random variable denoting the number of particles that particle i generates. Then

$$\begin{aligned} E\left(X_{n+1}^2 \middle| X_n = x\right) &= E\left((\xi_1 + \ldots + \xi_x)^2\right) \\ &= \sum_{1 \leq i, j \leq x} E\left(\xi_i \xi_j\right) \\ &= \sum_i E\left(\xi_i^2\right) + 2 \sum_{1 \leq i < j \leq x} E\left(\xi_i\right) E\left(\xi_j\right) \\ &= x(\sigma^2 + \mu^2) + 2\binom{x}{2}\mu^2 = x\sigma^2 + x^2\mu^2 \end{aligned}$$

Here $E\left(\xi_i \xi_j\right) = E\left(\xi_i\right) E\left(\xi_j\right)$ for $i \neq j$ due to the independence of the random variables.

Note

Alternatively, $E\left(X_{n+1}^2 \middle| X_n = x\right) = \operatorname{Var}\left(X_{n+1} | X_n = x\right) + E\left(X_{n+1} | X_n = x\right)^2 = \operatorname{Var}\left(\xi_1 + \ldots + \xi_x\right) + E\left(\xi_1 + \ldots + \xi_x\right)^2 = \sum \operatorname{Var}\left(\xi_i\right) + \left(\sum E\left(\xi_i\right)\right)^2 = x\sigma^2 + (x\mu)^2$ as $\xi_1, \ldots, \xi_x$ are independent and identically distributed.

2 Optional Part

17. Show that if x leads to y and y leads to z, then x leads to z

Solution: Since x leads to y and y leads to z, there exist $n, m \in \mathbb{Z}^+$ such that $P^n(x,y) > 0$, $P^m(y,z) > 0$. Then $\rho_{xz} \geq P^{n+m}(x,z) \geq P^m(y,z)P^n(x,y) > 0$, so x leads to z.

23. A certain Markov chain that arises in genetics has states $0, 1, \ldots, 2d$ and transition function

$$P(x,y) = \binom{2d}{y}\left(\frac{x}{2d}\right)^y\left(1-\frac{x}{2d}\right)^{2d-y}$$

Find $\rho_{\{0\}}(x)$, $0 < x < 2d$

Solution: Note that by the transition function, $P(0,0) = P(2d,2d) = 1$. So $0, 2d$ are absorbing and thus recurrent, and we can see that all other states are transient. In particular, $\rho_{\{0\}}(x) = \rho_{x0}$.

Let X_n, $n \geq 0$ be the chain random variables. By direct computation,

$$\begin{aligned}
E\left(X_{n+1}|X_n = x\right) &= \sum_{y=0}^{2d} yP(x,y) \\
&= \sum_{y=0}^{2d} y\binom{2d}{y}(\frac{x}{2d})^y(1-\frac{x}{2d})^{2d-y} \\
&= x\sum_{y=1}^{2d} \frac{(2d-1)!}{(y-1)!(2d-y)!}(\frac{x}{2d})^{y-1}(1-\frac{x}{2d})^{2d-y} \\
&= x
\end{aligned}$$

and so for $k \geq 0$

$$\begin{aligned}
E\left(X_{n+k+1}|X_n = x\right) &= \sum_{y=0}^{2d} E\left(X_{n+k+1}|X_n = x, X_{n+k} = y\right)P\left(X_{n+k} = y|X_n = x\right) \\
&= \sum_{y=0}^{2d} yP\left(X_{n+k} = y|X_n = x\right) \\
&= E\left(X_{n+k}|X_n = x\right)
\end{aligned}$$

Trivially, $E_x(X_0) = x$. By induction, $E_x(X_n) = E\left(X_n|X_0 = x\right) = x$ for all x and $n \geq 0$, which is a constant independent of n.

Since all states $y \in \{1, 2, \ldots, 2d-1\}$ are transient, we have $E_x(N(y)) = \sum_{n=1}^{\infty} P^n(x,y) < \infty$. This implies that $\lim_{n\to\infty} P^n(x,y) = 0$ for all such y.

As $0, 2d$ are absorbing, for $y \in \{0, 2d\}$ we must have $\lim_{n\to\infty} P^n(x,y) = \lim_{n\to\infty} P_x(T_y \leq n) = P_x(T_y < \infty) = \rho_{xy}$.

Combined, these imply that for all $x \in \{1, 2, \ldots, 2d-1\}$,

$$2d - x = \lim_{n\to\infty} E_x(2d - X_n) = \lim_{n\to\infty} \sum_{y=0}^{2d}(2d-y)P^n(x,y) = \sum_{y=0}^{2d}(2d-y)\lim_{n\to\infty} P^n(x,y) = 2d\rho_{x0}$$

and so $\rho_{x0} = \frac{2d-x}{2d} = 1 - \frac{x}{2d}$.

Note

The model in question is the Wright–Fisher model that describes the number of a specific type of allele at a given locus on N diploids (or equivalently $2N$ haploids) in generations undergoing binomial sampling, assuming there is only two allelic types. Please only consult the School of Life Sciences for what this means.

As shown in the proof, the chain is also a martingale. That $E_x(X_n) = x$ for all n is a result of this. The same proof can also be used on gambler's ruin chain with fair probability, which is why they have the same extinct probability.

If you are able to guess the form of the solution, you can verify it by checking the one-step argument equations

$$\begin{aligned}\rho_{x0} &= P(x,0) + \sum_{y=1}^{2d-1} P(x,y)\rho_{y0} \\ &= (1-\frac{x}{2d})^{2d} + \sum_{y=1}^{2d-1} \binom{2d}{y} (\frac{x}{2d})^y (1-\frac{x}{2d})^{2d-y} \rho_{y0}, \quad x \in \{1,\dots,2d-1\}\end{aligned}$$

As covered in lecture, the solution to this system exists and is unique since there are only finitely many (transient) states. So the solution you guessed must then be the correct one. For this question, the solution can be guessed by observing that, with $\rho_{00} = 1$ and $\rho_{2d,0} = 0$,

$$\begin{aligned}\rho_{x0} &= \sum_{y=0}^{2d} \binom{2d}{y} (\frac{x}{2d})^y (1-\frac{x}{2d})^{2d-y} \rho_{y0} \\ &= (1-\frac{x}{2d}) \sum_{y=0}^{2d-1} \frac{(2d-1)!}{y!(2d-1-y)!} (\frac{x}{2d})^y (1-\frac{x}{2d})^{2d-1-y} \cdot \frac{2d}{2d-y}\rho_{y0}\end{aligned}$$

and so if $\frac{2d}{2d-y}\rho_{y0} = 1$, the equation simplifies to $\rho_{x0} = 1 - \frac{x}{2d}$, and everything matches nicely.

If you have another solution, particularly if you are able to solve the one-step system above without guessing the solution, please share it with us.

25. A gambler playing roulette makes a series of one dollar bets. He has respective probability 9/19 and 10/19 of winning and losing each bet. The gambler decides to quit playing as soon as he either is one dollar ahead or has lost of initial capital of \$100.

 (a) Find the probability that when he quits playing he will have lost \$1000.

 (b) Find his expected loss.

Solution:

(a) By Q24, $P_{1000}(T_0 < T_{1001}) = \frac{(q/p)^x - (q/p)^d}{1-(q/p)^d} = (10/9)^{1000} \frac{10/9-1}{(10/9)^{1001}-1} \approx 0.1 + 1.6 \times 10^{-47}$ with $p = \frac{9}{19}$, $q = \frac{10}{19}$, $x = 1000$, and $d = 1001$.

Note

We can estimate this by $P_{1000}(T_0 < T_{1001}) \approx (10/9)^{1000} \frac{10/9-1}{(10/9)^{1001}} = \frac{1/9}{10/9} = \frac{1}{10}$, which is quite accurate.

(b) The expected *loss* is

$$\$1000 \cdot P_{1000}(T_0 < T_{1001}) - (\$1001 - \$1000)(1 - P_{1000}(T_0 < T_{1001})) = \$1001 \cdot P_{1000}(T_0 < T_{1001}) - \$1 \approx \$99.1$$

28. Consider an irreducible birth and death chain on the nonnegative integers. Show that if $p_x \le q_x$ for $x \ge 1$, the chain is recurrent.

Solution: For $y \in \mathbb{N}$, $\gamma_y = \prod_{x=1}^{y} \frac{q_x}{p_x} \ge \prod_{x=1}^{y} 1 = 1$. So $\sum_{y=0}^{\infty} \gamma_y \ge \sum_{y=0}^{\infty} 1 = \infty$. By Q26(a), $\rho_{10} = 1$.

As the chain is a birth and death chain on nonnegative numbers, $P(0,0) + P(0,1) = 1$. So by one-step argument,

$$\rho_{00} = P(0,0) + P(0,1)\rho_{10} = P(0,0) + P(0,1) = 1$$

Hence 0 is a recurrent state. As the chain is irreducible, the whole chain is recurrent.

Note

The converse does not hold: consider a chain with $p_1 = 2/3$, $q_1 = 1/3$, $p_x = \frac{x}{2x-1}$, $q_x = \frac{x-1}{2x-1}$ for $x \geq 2$. Then $p_x > q_x$ on $x \geq 1$ but $\gamma_y = \prod_{x=1}^{y} \frac{q_x}{p_x} = \frac{1}{2y}$ on $y \geq 1$ which gives $\sum_{y=0}^{\infty} \gamma_y = \infty$ and so the chain is recurrent.

30. Consider the birth and death chain in Example 13.
    (a) Compute $P_x(T_a < T_b)$ for $a < x < b$
    (b) Compute ρ_{x0}, $x > 0$

Solution: Recall that the chain has transition probabilities $p_x = \frac{x+2}{2(x+1)}$ and $q_x = \frac{x}{2(x+1)}$ for $x \geq 0$. As computed in the textbook, $\gamma_x = 2(\frac{1}{x+1} - \frac{1}{x+2})$ for $x \in \mathbb{N}$.

For $c < d$, $\sum_{x=c}^{d-1} \gamma_x = 2\sum_{x=c}^{d-1}(\frac{1}{x+1} - \frac{1}{x+2}) = 2(\frac{1}{c+1} - \frac{1}{d+1})$. In particular, $\sum_{x=c}^{\infty} \gamma_x = \frac{2}{c+1}$ and $\sum_{y=0}^{\infty} \gamma_y = 2 < \infty$.

(a) For $a < x < b$, $P_x(T_a < T_b) = \frac{\sum_{y=x}^{b-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y} = \frac{2(\frac{1}{x+1} - \frac{1}{b+1})}{2(\frac{1}{a+1} - \frac{1}{b+1})} = \frac{(b-x)(a+1)}{(b-a)(x+1)}$

(b) By Q26(b), for $x \geq 1$ we have $\rho_{x0} = \frac{\sum_{y=x}^{\infty} \gamma_y}{\sum_{y=0}^{\infty} \gamma_y} = \frac{2/(x+1)}{2} = \frac{1}{x+1}$

31. Consider a branching chain such that $f(1) < 1$. Show that every state other than 0 is transient

Solution: Note that 0 is an absorbing state. To analyze the recurrent probability, we will consider the following two cases:

- If $f(0) > 0$, then $\rho_{x0} \geq P(x, 0) = f(0)^x > 0$ for each $x \geq 1$. This implies that $\rho_{xx} < 1$ and so x is transient for all $x \geq 1$.
- If $f(0) = 0$, then for each $x \geq 1$ and $x > y \geq 0$, $P(x, y) = 0$, which implies that $\rho_{xx} = P(x, x) = f(1)^x < 1$ and thus x is transient.

33. Consider a branching chain with $f(0) = f(3) = 1/2$. Find the probability ρ of extinction

Solution: The expected number of particles generated is $E(\xi) = 0 \cdot f(0) + 3 \cdot f(3) = 3/2 > 1$. So we need to consider the generating function Φ of the distribution of offspring.

The generating function is $\Phi(t) = \sum f(n)t^n = \frac{1}{2}(1 + t^3)$, and so $\Phi(t) = t$, or equivalently $(t-1)(t^2 + t - 1) = 0$, has solutions $t = 1$, $t = \frac{-1+\sqrt{5}}{2} \in [0, 1]$, $t = \frac{-1-\sqrt{5}}{2} < 0$. Hence the extinction probability is $\rho = \frac{-1+\sqrt{5}}{2} \approx 0.618$

35. Let X_n, $n \geq 0$, be a branching chain. Show that $E_x(X_n) = x\mu^n$

Solution: We will show by induction on n that $E_x(X_n) = x\mu^n$ for each $n \geq 0$. The base cases $n = 0$ and $n = 1$ hold trivially. Suppose now that $E_x(X_k) = x\mu^k$ for some $k \geq 1$. Then by the law of total expectation,

$$\begin{aligned}
E_x(X_{k+1}) = E(X_{k+1}|X_0 = x) &= \sum_{y=0}^{\infty} E(X_{k+1}|X_k = y,\, X_0 = x)\, P(X_k = y|X_0 = x) \\
&= \sum_{y=0}^{\infty} E(X_{k+1}|X_k = y)\, P(X_k = y|X_0 = x) \\
&= \sum_{y=0}^{\infty} y\mu P(X_k = y|X_0 = x) \\
&= \mu E(X_k|X_0 = x) \\
&= x\mu^{k+1}
\end{aligned}$$

So by induction $E_x(X_n) = x\mu^n$ for all $n \geq 0$.

36(b, c, d). Let X_n, $n \geq 0$, be a branching chain and suppose that the associated random variable ξ has finite variance σ^2.

(b) Use Exercise 35 to show that

$$E_x(X_{n+1}^2) = x\mu^n\sigma^2 + \mu^2 E_x(X_n^2)$$

(c) Show that

$$E_x(X_n^2) = x\sigma^2(\mu^{n-1} + \ldots + \mu^{2(n-1)}) + x^2\mu^{2n}, \quad n \geq 1$$

(d) Show that if there are x particles initially, then for $n \geq 1$,

$$\mathrm{Var}\,(X_n) = \begin{cases} x\sigma^2\mu^{n-1}\left(\frac{1-\mu^n}{1-\mu}\right) & \mu \neq 1 \\ nx\sigma^2 & \mu = 1 \end{cases}$$

Solution:

(b) By part (a) and Q35,

$$\begin{aligned} E_x(X_{n+1}^2) &= \sum_{y=0}^{\infty} E\left(X_{n+1}^2 \middle| X_n = y\right) P_x(X_n = y) \\ &= \sum_{y=0}^{\infty}(y\sigma^2 + y^2\mu^2)P_x(X_n = y) \\ &= \sigma^2 E_x(X_n) + \mu^2 E_x(X_n^2) \\ &= \sigma^2 x\mu^n + \mu^2 E_x(X_n^2) \end{aligned}$$

(c) By part (a), $E_x(X_1^2) = E\left(X_1^2 \middle| X_0 = x\right) = x\sigma^2 + x^2\mu^2$.
If $\mu = 1$, then by part (b) $E_x(X_{n+1}^2) = x\sigma^2 + E_x(X_n^2)$ and so $E_x(X_n^2) = nx\sigma^2 + x^2 = x\sigma^2(\mu^{n-1} + \ldots + \mu^{2n-2}) + x^2\mu^{2n}$.
Consider now the case that $\mu < 1$. By the result of part (b),

$$\begin{aligned} E_x(X_{n+1}^2) &= x\mu^n\sigma^2 + \mu^2 E_x(X_n^2) \\ \mu^{1-(n+1)}E_x(X_{n+1}^2) - \frac{x\sigma^2}{1-\mu} &= \mu\left(\mu^{1-n}E_x(X_n^2) - \frac{x\sigma^2}{1-\mu}\right) \\ \text{which implies} \quad \mu^{1-n}E_x(X_n^2) - \frac{x\sigma^2}{1-\mu} &= \mu^{n-1}\left(E_x(X_1^2) - \frac{x\sigma^2}{1-\mu}\right) \\ &= \mu^{n-1}\left(-x\sigma^2\frac{\mu}{1-\mu} + x^2\mu^2\right) \\ E_x(X_n^2) &= x\sigma^2\mu^{n-1}\frac{1-\mu^n}{1-\mu} + x^2\mu^{2n} \\ &= x\sigma^2(\mu^{n-1} + \ldots + \mu^{2n-2}) + x^2\mu^{2n} \end{aligned}$$

Note

Alternatively this can also be done by induction.

(d) By Q35, $E_x(X_n) = x\mu^n$. So

$$\begin{aligned} \mathrm{Var}\,(X_n) &= E_x(X_n^2) - E_x(X_n)^2 \\ &= x\sigma^2(\mu^{n-1} + \ldots + \mu^{2n-2}) + x^2\mu^{2n} - (x\mu^n)^2 \\ &= x\sigma^2(\mu^{n-1} + \ldots + \mu^{2n-2}) \\ &= \begin{cases} x\sigma^2\mu^{n-1}\frac{1-\mu^n}{1-\mu} & \text{if } \mu \neq 1 \\ nx\sigma^2 & \text{if } \mu = 1 \end{cases} \end{aligned}$$

37. Consider the queuing chain
   (a) Show that if either $f(0) = 0$ or $f(0) + f(1) = 1$, the chain is not irreducible
   (b) Show that $f(0) > 0$ and $f(0) + f(1) < 1$, the chain is irreducible

Solution: Recall that the transition function is

$$P(x, y) = \begin{cases} f(y) & \text{if } x = 0 \\ f(y - x + 1) & \text{otherwise} \end{cases}$$

for $y \geq x - 1$ and $y \geq 0$.

(a) If $f(0) = 0$, then for $x \geq 1$, $P(x, x-1) = f(0) = 0$ and so $P(X_{n+1} < X_n) = 0$. This implies that x does not lead to $x - 1$ for all $x \geq 1$, and thus the chain is not irreducible.

If $f(0) + f(1) = 1$, then $P(x, x+k) = f(k+1) = 0$ for each $k \geq 1$, so $P(X_{n+1} > X_n) = 0$. This implies that x does not leads to $x + 1$ for all $x \geq 0$, and thus the chain is not irreducible.

(b) Suppose $f(0) > 0$ and $f(0) + f(1) < 1$. Then there exists $k \geq 2$ such that $f(k) > 0$.

Let $x \geq 0$. Then

- $\rho_{00} \geq P(0, 0) = f(0) > 0$, so 0 leads to 0
- if $x > 0$, $\rho_{x0} \geq P(x, x-1)P(x-1, x-2) \ldots P(1, 0) = f(0)^x > 0$, so x leads to 0
- With $n \geq \frac{x-1}{k-1}$ sufficiently large,

$$\begin{aligned} \rho_{0x} &\geq P(0,k)P(k,2k-1)P(2k-1,3k-2)\ldots P((n-1)(k-1)+1, n(k-1)+1) \\ &\quad \times P(n(k-1)+1, n(k-1)) \ldots P(x+1, x) \\ &= f(k)^n f(0)^{n(k-1)-x+1} > 0 \end{aligned}$$

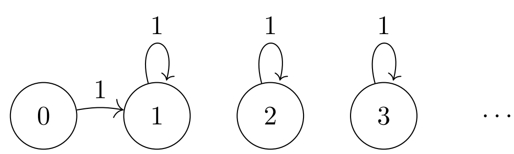
  so 0 leads to x

As x is arbitrary, every state leads to every other state, and thus the chain is irreducible.

38. Determine which states of the queuing chain are absorbing, which are recurrent, and which are transient, when the chain is not irreducible. Consider the following four cases separately:
   (a) $f(1) = 1$
   (b) $f(0) > 0$, $f(1) > 0$, and $f(0) + f(1) = 1$
   (c) $f(0) = 1$
   (d) $f(0) = 0$ and $f(1) < 1$

Solution:

(a) Since $f(1) = 1$, the transition diagram is then

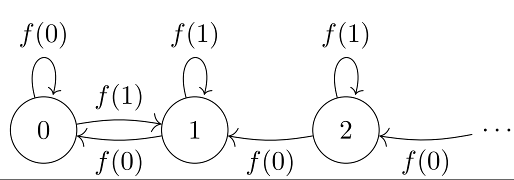


Easy to see that for each $x \geq 1$, x is an absorbing state and thus a recurrent state.

Since $f(1) = 1$, we must have $f(0) = 0$. So $\rho_{00} = P(0, 0) = f(0) = 0 < 1$ and thus 0 is a transient state

By Q37(a), the chain is not irreducible.

(b) The transition diagram is



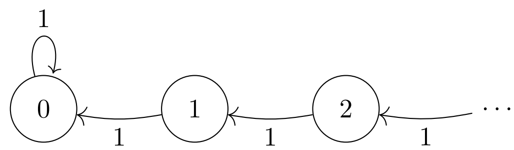
As no state satisfies $P(x,x) = 1$, the chain has no absorbing state.

As $\{0,1\}$ is irreducible closed, $0,1$ are recurrent.

For each $x \geq 2$, $\rho_{xx} \leq 1 - \rho_{\{0,1\}}(x) = 1 - P(x,x-1)\ldots P(2,1) = 1 - f(0)^{x-1} < 1$, so x is transient.

Since $f(0) + f(1) = 1$, by Q37(a) the chain is not irreducible.

(c) The transition diagram is



As $P(0,0) = 1$, 0 is absorbing and thus recurrent. For $x \geq 1$, $\rho_{xx} \leq 1 - \rho_{\{0\}}(x) = 1 - P(x,x-1)\ldots P(1,0) = 1 - 1^x = 0$, so x is transient.

Since $f(0) = 1$, we have have $f(0) + f(1) = 1$, by Q37(a) the chain is not irreducible.

(d) The transition function is $P(x,y) = \begin{cases} f(y) & \text{if } x = 0 \\ f(y-x+1) & \text{if } x \geq 1 \end{cases}$ for $y \geq x \geq 0$

As no state satisfies $P(x,x) = 1$, the chain has no absorbing state.

As argued in Q37(a), $\rho_{xy} = 0$ for $y < x$. Furthermore, as $f(0) + f(1) < 1$, there exists $k \geq 2$ with $f(k) > 0$. This implies that $\rho_{00} \leq 1 - P(0,1) = 1 - f(1) < 1$, and for each $x \geq 0$, $\rho_{xx} \leq 1 - P(x, x+k-1) = 1 - f(k) < 1$. So every state is transient.

Since $f(0) = 0$, by Q37(a) the chain is not irreducible.