# MATH4240 Homework 2 Reference Solution

## 1 Compulsory Part

1. Let $X_n$, $n \geq 0$ be the two-state Markov chain. Find
   (a) $P\left(\, X_1 = 0 \mid X_0 = 0 \text{ and } X_2 = 0 \,\right)$
   (b) $P\left(X_1 \neq X_2\right)$

**Solution:** The transition matrix for such chain is

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

with $p, q \in [0,1]$.

(a)

$$\begin{aligned}
&P\left(\, X_1 = 0 \mid X_0 = 0,\, X_2 = 0 \,\right) \\
=& \frac{P\left(X_1 = 0,\, X_0 = 0,\, X_2 = 0\right)}{P\left(X_0 = 0,\, X_2 = 0\right)} \\
=& \frac{P\left(X_1 = 0,\, X_0 = 0,\, X_2 = 0\right)}{P\left(X_1 = 0,\, X_0 = 0,\, X_2 = 0\right) + P\left(X_1 = 1,\, X_0 = 0,\, X_2 = 0\right)} \\
=& \frac{P\left(X_2 = 0 | X_1 = 0\right) P\left(X_1 = 0 | X_0 = 0\right) P\left(X_0 = 0\right)}{P\left(X_2 = 0 | X_1 = 0\right) P\left(X_1 = 0 | X_0 = 0\right) P\left(X_0 = 0\right) + P\left(X_2 = 0 | X_1 = 1\right) P\left(X_1 = 1 | X_0 = 0\right) P\left(X_0 = 0\right)} \\
=& \frac{P(0,0)P(0,0)\pi_0(0)}{P(0,0)P(0,0)\pi_0(0) + P(1,0)P(0,1)\pi_0(0)} \\
=& \frac{(1-p)^2}{(1-p)^2 + pq}
\end{aligned}$$

### Note

Note that $(1-p)^2 + pq = P^2(0,0)$ and $(1-p)^2 = P(0,0)^2$

(b) We first compute $\pi_1$.

$$\pi_1 = \pi_0 P = (\pi_0(0), \pi_0(1)) \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} = ((1-p)\pi_0(0) + q\pi_0(1),\, p\pi_0(0) + (1-q)\pi_0(1))$$

Then,

$$\begin{aligned}
&P\left(X_1 \neq X_2\right) \\
&= P\left(X_2 = 1,\, X_1 = 0\right) + P\left(X_2 = 0,\, X_1 = 1\right) \\
&= P\left(X_2 = 1 | X_1 = 0\right) P\left(X_1 = 0\right) + P\left(X_2 = 0 | X_1 = 1\right) P\left(X_1 = 1\right) \\
&= P(0,1)\pi_1(0) + P(1,0)\pi_1(1) \\
&= pq + \pi_0(0)p(1-p) + \pi_0(1)q(1-q)
\end{aligned}$$

2. Suppose we have two boxes and $2d$ balls, of which $d$ are black and $d$ are red. Initially, $d$ of the balls are placed in box 1, and the remainder of the balls are placed in box 2. At each trial a ball is chosen at random from each of the boxes, and the two balls are put back in the opposite boxes. Let $X_0$ denote the number of black balls initially in box 1 and, for

$n \geq 1$, let $X_n$ denote the number of black balls in box 1 after the nth trial. Find the transition function of the Markov chain $X_n$ , $n \geq 0$.

**Solution:** Note that the state space is $\mathscr{S} = \{ 0, 1, \ldots, d \}$. Recall that we want to find $P(x,y) = P(X_{n+1} = y | X_n = x)$
Easy to see that

- if $x = 0$, then all balls in box 2 are black, and so $P(0,y) = \begin{cases} 1 & \text{if } y = 1 \\ 0 & \text{otherwise} \end{cases}$
- if $x = d$, then all balls in box 2 are red, and so $P(d,y) = \begin{cases} 1 & \text{if } y = d-1 \\ 0 & \text{otherwise} \end{cases}$

If $x \in \{ 1, \ldots, d-1 \}$, then box 1 has $x$ black balls and box 2 has $d-x$ black balls. Enumerating all 4 cases of colors of balls chosen:

- Black from box 1, black from box 2: this happens at probability $\frac{x}{d}\frac{d-x}{d}$ and gives $y = x$
- Black from box 1, red from box 2: this happens at probability $\frac{x}{d}\frac{x}{d}$ and gives $y = x-1$
- Red from box 1, black from box 2: this happens at probability $\frac{d-x}{d}\frac{d-x}{d}$ and gives $y = x+1$
- Red from box 1, red from box 2: this happens at probability $\frac{d-x}{d}\frac{x}{d}$ and gives $y = x$

which gives

$$P(x,y) = \begin{cases} (\frac{x}{d})^2 & \text{if } y = x-1 \\ 2\frac{x}{d}(1-\frac{x}{d}) & \text{if } y = x \\ (1-\frac{x}{d})^2 & \text{if } y = x+1 \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in \mathscr{S}$.

3. Let the queuing chain be modified by supposing that if there are one or more customers waiting to be served at the start of a period, there is probability $p$ that one customer will be served during that period and probability $1-p$ that no customers will be served during that period. Find the transition function for this modified queuing chain.

**Solution:** Similar to the queuing chain, let $X_0$ denote the number of customers present initially, $X_n$ denote the number of customers present at the end of the $n$th period for $n \geq 1$, $\xi_n$ denote the number of new customers arriving during the $n$th period, and assume that $\xi_1, \xi_2, \ldots$ are independent nonnegative integer-valued and have common density $f$ with $f(m) = 0$ for $m < 0$. This implies that the state space is $\mathscr{S} = \mathbb{N} = \{0, 1, \ldots\}$.
If $X_n = 0$, then $X_{n+1} = \xi_{n+1}$ and so $P(0,y) = f(y)$.
If $X_n \geq 1$, then $X_{n+1} = \begin{cases} X_n + \xi_{n+1} - 1 & \text{with probability } p \\ X_n + \xi_{n+1} & \text{with probability } 1-p \end{cases}$ and so

$$\begin{aligned} P(x,y) = P(X_{n+1} = y | X_n = x) &= pP(\xi_{n+1} = y - x + 1) + (1-p)P(\xi_{n+1} = y - x) \\ &= pf(y-x+1) + (1-p)f(y-x) \end{aligned}$$

on $x \geq 1$.

Hence the transition function is $P(x,y) = \begin{cases} f(y) & \text{if } x = 0 \\ pf(y-x+1) + (1-p)f(y-x) & \text{if } x \geq 1 \end{cases}$ for $x, y \in \mathscr{S}$.

## Note

We pose no assumption on the arrival rate, so the queue is G/G/1 and not M/G/1.

5. Let $X_n$, $n \geq 0$ be the two-state Markov chain.

(a) Find $P_0(T_0 = n)$

(b) Find $P_0(T_1 = n)$

**Solution:**

(a) Consider first the case $n = 1$. Then $P_0(T_0 = 1) = P(X_1 = 0 | X_0 = 0) = 1 - p$.

For $n \geq 2$, $T_0 = n$ when starting on state 0 if and only if $X_1 = \ldots = X_{n-1} = 1$ and $X_n = 0$. This implies that $P_0(T_0 = n) = P(X_1 = \ldots = X_{n-1} = 1, X_n = 0 | X_0 = 0) = P(0,1)P(1,1)^{n-2}P(1,0) = pq(1-q)^{n-2}$

Combined this gives

$$P_0(T_0 = n) = \begin{cases} 1 - p & \text{if } n = 1 \\ pq(1-p)^{n-2} & \text{otherwise} \end{cases}$$

(b) When starting on state 0, $T_1 = n$ if and only if $X_1 = \ldots = X_{n-1} = 0$ and $X_n = 1$, so $P_0(T_1 = n) = P(X_1 = \ldots = X_{n-1} = 0, X_n = 1 | X_0 = 0) = (1-p)^{n-1}p$

10. Consider the Ehrenfest chain with $d = 3$

(a) Find $P_x(T_0 = n)$ for $x \in \mathscr{S}$ and $1 \leq n \leq 3$

(b) Find $P$, $P^2$, and $P^3$

(c) Let $\pi_0$ be the uniform distribution $\pi_0 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Find $\pi_1$, $\pi_2$ and $\pi_3$

**Solution:** By definition, the transition matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(a) Observe that $X_n$ must have the same parity as $X_0 + n$ and $| X_0 - X_n | \leq n$. This implies that $P_0(T_0 = 1), P_0(T_0 = 3), P_1(T_0 = 2), P_2(T_0 = 1), P_2(T_0 = 3), P_3(T_0 = 1), P_3(T_0 = 2)$ are all zero. The remaining are

- $P_0(T_0 = 2) = P(0,1)P(1,0) = 1/3$
- $P_1(T_0 = 1) = P(1,0) = 1/3$
- $P_1(T_0 = 3) = P(1,2)P(2,1)P(1,0) = 4/27$
- $P_2(T_0 = 2) = P(2,1)P(1,0) = 2/9$
- $P_3(T_0 = 3) = P(3,2)P(2,1)P(1,0) = 2/9$

### Note

$E_x(T_0 = n)$

| x \ n | 1 | 2 | 3 |
|---|---|---|---|
| 0 | 0 | 1/3 | 0 |
| 1 | 1/3 | 0 | 4/27 |
| 2 | 0 | 2/9 | 0 |
| 3 | 0 | 0 | 2/9 |

(b) By direct computation,

$$P^2 = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 \\ 0 & 7/9 & 0 & 2/9 \\ 2/9 & 0 & 7/9 & 0 \\ 0 & 2/3 & 0 & 1/3 \end{pmatrix}, \quad P^3 = \begin{pmatrix} 0 & 7/9 & 0 & 2/9 \\ 7/27 & 0 & 20/27 & 0 \\ 0 & 20/27 & 0 & 7/27 \\ 2/9 & 0 & 7/9 & 0 \end{pmatrix}$$

(c)

$$\pi_1 = \pi_0 P = (\frac{1}{12}, \frac{5}{12}, \frac{5}{12}, \frac{1}{12})$$
$$\pi_2 = \pi_1 P = (\frac{5}{36}, \frac{13}{36}, \frac{13}{36}, \frac{5}{36})$$
$$\pi_3 = \pi_2 P = (\frac{13}{108}, \frac{41}{108}, \frac{41}{108}, \frac{13}{108})$$

19. Consider a Markov chain having state space $\{ 0, 1, \ldots, 6 \}$ and transition matrix

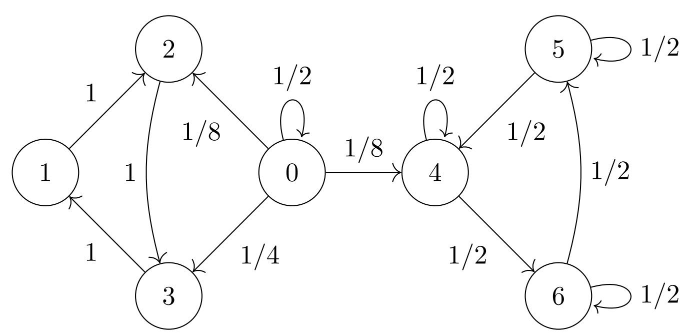
$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

(a) Determine which states are transient and which states are recurrent

(b) Find $\rho_{0y}$, $y = 0, \ldots, 6$

**Solution:**

(a) We first draw the state transition diagram:




It is now easy to see that $C_1 = \{ 1, 2, 3 \}$ and $C_2 = \{ 4, 5, 6 \}$ are irreducible closed, and 0 is transient and $1, 2, 3, 4, 5, 6$ are recurrent.

(b) Since neither of $2, 3, 4$ leads to 0, $\rho_{00} = P(0, 0) = 1/2$.

As $C_1, C_2$ are irreducible closed sets of recurrent states, $\rho_{0x} = \rho_{C_1}(0)$ for each $x \in C_1$, and $\rho_{0y} = \rho_{C_2}(0)$ for each $y \in C_2$. Note that

$$\rho_{C_1}(0) = \sum_{y \in C_1} P(0, y) + P(0, 0)\rho_{C_1}(0) = \frac{3}{8} + \frac{1}{2}\rho_{C_1}(0)$$
$$\rho_{C_2}(0) = \sum_{y \in C_2} P(0, y) + P(0, 0)\rho_{C_2}(0) = \frac{1}{8} + \frac{1}{2}\rho_{C_2}(0)$$

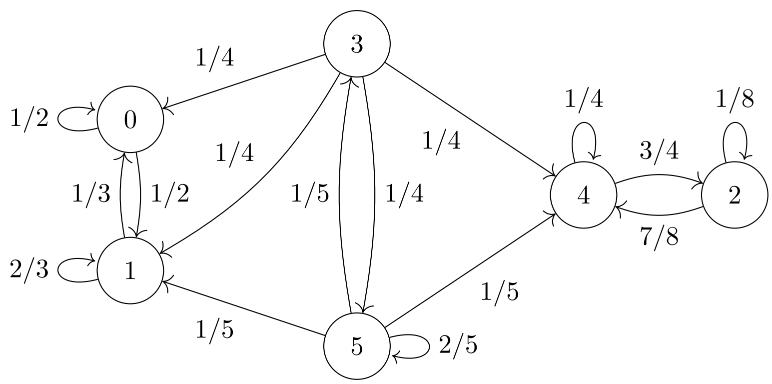
solving this we have $\rho_{01} = \rho_{02} = \rho_{03} = \rho_{C_1}(0) = 3/4$ and $\rho_{04} = \rho_{05} = \rho_{06} = \rho_{C_2}(0) = 1/4$.

20(a). Consider the Markov chain on $\{ 0, 1, \ldots, 5 \}$ having transition matrix

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix} \end{array}$$

Determine which states are transient and which are recurrent.

**Solution:** Let us draw the state transition diagram:




We can see that $C_1 = \{ 0, 1 \}$, $C_2 = \{ 2, 4 \}$ are irreducible closed, and thus $3, 5$ are transient and $0, 1, 2, 4$ are recurrent.

## 2 Optional Part

4. Consider a probability space $(\Omega, \mathcal{A}, P)$ and assume that the various sets mentioned below are all in $\mathcal{A}$.
   (a) Show that if $D_i$ are disjoint and $P(C|D_i) = p$ independently of $i$, then $P(\, C \mid \bigcup_i D_i \,) = p$
   (b) Show that if $C_i$ are disjoint, then $P(\, \bigcup_i C_i \mid D \,) = \sum_i P(C_i|D)$
   (c) Show that if $E_i$ are disjoint and $\bigcup_i E_i = \Omega$, then $P(C|D) = \sum_i P(E_i|D)\, P(C|E_i \cap D)$
   (d) Show that if $C_i$ are disjoint and $P(A|C_i) = P(B|C_i)$ for all $i$, then $P(\, A \mid \bigcup_i C_i \,) = P(\, B \mid \bigcup_i C_i \,)$

**Solution:**

(a)

$$P\left( C \,\middle|\, \bigcup_i D_i \right) = \frac{P(C \cap \bigcup_i D_i)}{P(\bigcup_i D_i)} = \frac{P(\bigcup_i (C \cap D_i))}{P(\bigcup_i D_i)} = \frac{\sum_i P(C \cap D_i)}{\sum_i P(D_i)}$$
$$= \frac{\sum_i P(C|D_i)\, P(D_i)}{\sum_i P(D_i)} = \frac{\sum_i pP(D_i)}{\sum_i P(D_i)} = p$$

(b)

$$P\left( \bigcup_i C_i \,\middle|\, D \right) = \frac{P((\bigcup_i C_i) \cap D)}{P(D)} = \frac{P(\bigcup_i (C_i \cap D))}{P(D)} = \frac{\sum_i P(C_i \cap D)}{P(D)} = \sum_i P(C_i|D)$$

(c)

$$P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{\sum_i P(C \cap D \cap E_i)}{P(D)} = \sum_i \frac{P(C \cap D \cap E_i)}{P(D \cap E_i)} \frac{P(D \cap E_i)}{P(D)} = \sum_i P(C|D \cap E_i)\, P(E_i|D)$$

(d)

$$P\left( A \,\middle|\, \bigcup_i C_i \right) = \frac{P(A \cap \bigcup_i C_i)}{P(\bigcup_i C_i)} = \frac{P(\bigcup_i (A \cap C_i))}{P(\bigcup_i C_i)} = \frac{\sum_i P(A|C_i)\, P(C_i)}{P(\bigcup_i C_i)}$$
$$= \frac{\sum_i P(B|C_i)\, P(C_i)}{P(\bigcup_i C_i)} = \frac{P(\bigcup_i (B \cap C_i))}{P(\bigcup_i C_i)} = \frac{P(B \cap \bigcup_i C_i)}{P(\bigcup_i C_i)} = P\left( B \,\middle|\, \bigcup_i C_i \right)$$

6. Let $X_n$, $n \geq 0$ be the Ehrenfest chain and suppose that $X_0$ has a binomial distribution with parameter $d$ and $1/2$, i.e. $P(X_0 = x) = \binom{d}{x}/2^d$, $x = 0, \ldots, d$. Find the distribution of $X_1$.

**Solution:** Recall that the transition function is

$$P(x, y) = \begin{cases} x/d & \text{if } y = x - 1 \\ 1 - x/d & \text{if } y = x + 1 \end{cases}$$

Noting that $P(X_1 = y) = \sum_x P(X_0 = x)\, P(x, y)$,

- On $y = 0$, $P(X_1 = 0) = P(1, 0) P(X_0 = 1) = \frac{1}{d} \cdot \binom{d}{1} 2^{-d} = 2^{-d} = \binom{d}{0} 2^{-d}$
- On $y = d$, $P(X_1 = d) = P(d-1, d) P(X_0 = d - 1) = (1 - \frac{d-1}{d}) \cdot \binom{d}{d-1} 2^{-d} = 2^{-d} = \binom{d}{d} 2^{-d}$
- On $1 \leq y \leq d - 1$,

$$\begin{aligned} P(X_1 = y) &= P(y-1, y) P(X_0 = y - 1) + P(y+1, y) P(X_0 = y + 1) \\ &= (1 - \frac{y-1}{d}) \binom{d}{y-1} 2^{-d} + \frac{y+1}{d} \binom{d}{y+1} 2^{-d} \\ &= 2^{-d} \left( \frac{d-y+1}{d} \frac{d!}{(y-1)!(d-y+1)!} + \frac{y+1}{d} \frac{d!}{(y+1)!(d-y-1)!} \right) \\ &= 2^{-d} (\binom{d-1}{y-1} + \binom{d-1}{y}) = \binom{d}{y} 2^{-d} \end{aligned}$$

Therefore $X_1$ is still of binomial distribution with parameter $d$ and $1/2$.

7. Let $X_n$, $n \geq 0$ be a Markov chain. Show that $P(X_0 = x_0 | X_1 = x_1, \ldots, x_n = x_n) = P(X_0 = x_0 | X_1 = x_1)$

**Solution:**

$$\begin{aligned} & P(X_0 = x_0 | X_1 = x_1, \ldots, X_n = x_n) \\ = & \frac{P(X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n)}{P(X_1 = x_1, \ldots, X_n = x_n)} \\ = & \frac{P(X_0 = x_0, X_1 = x_1) P(x_1, x_2) \ldots P(x_{n-1}, x_n)}{P(X_1 = x_1) P(x_1, x_2) \ldots P(x_{n-1}, x_n)} \\ = & \frac{P(X_0 = x_0, X_1 = x_1)}{P(X_1 = x_1)} \\ = & P(X_0 = x_0 | X_1 = x_1) \end{aligned}$$

8. Let $x$ and $y$ be distinct states of a Markov chain having $d < \infty$ states and supposes that $x$ leads to $y$. Let $n_0$ be the smallest positive integer such that $P^{n_0}(x, y) > 0$ and let $x_1, \ldots, x_{n_0-1}$ be such that

$$P(x, x_1)P(x_1, x_2) \ldots P(x_{n_0-2}, x_{n_0-1})P(x_{n_0-1}, y) > 0$$

(a) Show that $x, x_1, \ldots, x_{n_0-1}, y$ are distinct states
(b) Use (a) to show that $n_0 \leq d - 1$
(c) Conclude that $P_x(T_y \leq d - 1) > 0$

**Solution:**

(a) Denote $x_0 = x$, $x_{n_0} = y$. Suppose there exist $0 \leq i < j \leq n_0$ such that $x_i = x_j$, then

$$P^{n_0-(j-i)}(x, y) \geq P(x_0, x_1) \cdots P(x_{i-1}, x_i)P(x_j, x_{j+1}) \cdots P(x_{n_0-1}, x_{n_0}) \geq P(x_0, x_1) \cdots P(x_{n_0-1}, x_{n_0}) > 0$$

So $N = n_0 - (j - i) < n_0$ is a smaller integer such that $P^N(x, y) > 0$. Contradiction arises.

Hence $x_0 = x, x_1, \ldots, x_{n_0-1}, x_{n_0} = y$ are all distinct.

(b) By (a), the chain must have $n_0 + 1$ distinct states, so $n_0 + 1 \leq d$, or $n_0 \leq d - 1$

(c) By previous part, $n_0 \leq d - 1$, so $P_x(T_y \leq d - 1) \geq P_x(T_y \leq n_0) \geq P(x, x_1)P(x_1, x_2) \cdots P(x_{n_0-1}, y) > 0$

## Note

The same idea can be applied to prove pumping lemma for e.g. deterministic finite automata.

9. Use (29) to verify the following identities:
(a) $P_x(T_y \leq n + 1) = P(x, y) + \sum_{x \neq y} P(x, z)P_z(T_y \leq n)$, $n \geq 0$
(b) $\rho_{xy} = P(x, y) + \sum_{z \neq y} P(x, z)\rho_{zy}$

**Solution:**

(a)

$$\begin{aligned} P_x(T_y \leq n+1) = P_x(T_y = 1) + \sum_{k=1}^{n} P_x(T_y = k+1) = P(x, y) + \sum_{k=1}^{n} \sum_{z \neq y} P(x, z)P_z(T_y = k) \\ = P(x, y) + \sum_{z \neq y} P(x, z) \sum_{k=1}^{n} P_z(T_y = k) = P(x, y) + \sum_{z \neq y} P_z(T_y \leq n) \end{aligned}$$

Here interchanging the order of summations is justified (in the case of infinite states) as the summands are all nonnegative.

(b)

$$\begin{aligned}\rho_{xy} = P_x(T_y < \infty) &= \lim_{n\to\infty} P_x(T_y \le n+1) \\ &= P(x,y) + \lim_{n\to\infty} \sum_{z\neq y} P(x,z)P_z(T_y \le n) \\ &= P(x,y) + \sum_{z\neq y} P(x,z) \lim_{n\to\infty} P_z(T_y \le n) \\ &= P(x,y) + \sum_{z\neq y} P(x,z)\rho_{z,y}\end{aligned}$$

Here interchanging the order of limit and summation is justified (in the case of infinite states) as the summand $P(x,z)P_z(T_y \le n)$ is non-decreasing in $n$.

11. Consider the genetics chain from Example 7 with $d = 3$
    (a) Find the transition matrices $P$ and $P^2$
    (b) If $\pi_0 = (0, \frac{1}{2}, \frac{1}{2}, 0)$, find $\pi_1$ and $\pi_2$
    (c) Find $P_x(T_{\{0,3\}} = n)$, $x \in \mathscr{S}$ for $n = 1$ and $n = 2$

**Solution:** The transition function is $P(x,y) = \binom{2x}{y}\binom{2(d-x)}{d-y}/\binom{2d}{d} = \binom{2x}{y}\binom{6-2x}{3-y}/20$ for $x, y \in \mathscr{S} = \{ 0, 1, \ldots, d \} = \{ 0, 1, 2, 3 \}$.

(a) By direct computation,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/5 & 3/5 & 1/5 & 0 \\ 0 & 1/5 & 3/5 & 1/5 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 8/25 & 2/5 & 6/25 & 1/25 \\ 1/25 & 6/25 & 2/5 & 8/25 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) By direct computation,

$$\pi_1 = \pi_0 P = (\frac{1}{10}, \frac{2}{5}, \frac{2}{5}, \frac{1}{10})$$
$$\pi_2 = \pi_1 P = (\frac{9}{50}, \frac{8}{25}, \frac{8}{25}, \frac{9}{50})$$

(c) As $0, 3$ are absorbing states, $P_0(T_{\{0,3\}=1}) = P_3(T_{\{0,3\}=1}) = 1$ and $P_0(T_{\{0,3\}=2}) = P_3(T_{\{0,3\}=2}) = 0$.

For $x = 1, 2$,

- $P_1(T_{\{0,3\}} = 1) = P(1,0) = 1/5$
- $P_1(T_{\{0,3\}} = 2) = P(1,1)P(1,0) + P(1,2)P(2,3) = 4/25$
- $P_2(T_{\{0,3\}} = 1) = P(2,3) = 1/5$
- $P_2(T_{\{0,3\}} = 2) = P(2,1)P(1,0) + P(2,2)P(2,3) = 4/25$

12. Consider the Markov chain having state space $\{ 0, 1, 2 \}$ and transition matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1-p & 0 & p \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

(a) Find $P^2$
(b) Show that $P^4 = P^2$

(c) Find $P^n$, $n \geq 1$

**Solution:**

(a) By direct computation, $P^2 = \begin{pmatrix} 1-p & 0 & p \\ 0 & 1 & 0 \\ 1-p & 0 & p \end{pmatrix}$

(b) By direct computation, $P^4 = (P^2)^2 = \begin{pmatrix} 1-p & 0 & p \\ 0 & 1 & 0 \\ 1-p & 0 & p \end{pmatrix} = P^2$

(c) Since $(P^2)^2 = P^2$, we have

- when $n = 2k$, $k \in \mathbb{Z}^+$, $P^n = P^{2k} = (P^2)^k = P^2 = \begin{pmatrix} 1-p & 0 & p \\ 0 & 1 & 0 \\ 1-p & 0 & p \end{pmatrix}$
- By direct computation, $P^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1-p & 0 & p \\ 0 & 1 & 0 \end{pmatrix} = P$. So when $n = 2k+1$, $k \in \mathbb{Z}^+$, $P^n = P^{2k}P = P^3 = P$

Hence $P^n = \begin{cases} P^2 & \text{if } n \text{ is even} \\ P & \text{if } n \text{ is odd} \end{cases}$ for $n \geq 1$.

## Note

You can also see that $P^3 = P$ by observing that $P$ is $3 \times 3$ matrix and $P^2 \neq I$.