MATH4240 Homework 1 Reference Solution

1. Let X and Y have the joint probability density function given by

$$f_{X,Y}(x,y) = \begin{cases} 6(1-y) & \text{if } 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal density function $f_X(x)$ for X

(b) Find the conditional density function of X given $Y = y$

(c) Are X and Y independent? Explain why or why not

(d) Find $P\left(Y \ge \frac{3}{4} \middle| X = \frac{1}{2}\right)$

(e) Find $\mathrm{E}\left(X - 3Y\right)$

Solution:

(a)

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ \mathrm{d}y = \int_{-\infty}^{\infty} 6(1-y)\chi_{0\le x\le y\le 1}\ \mathrm{d}y \\ &= \chi_{0\le x\le 1}\int_{-\infty}^{\infty} 6(1-y)\chi_{x\le y\le 1}\ \mathrm{d}y = \chi_{0\le x\le 1}\int_{x}^{1} 6(1-y)\ \mathrm{d}y \\ &= 3(1-x)^2\chi_{0\le x\le 1} \end{aligned}$$

(b) To find the conditional density of X given $Y = y$, we first compute the marginal density f_Y for Y. With the same approach as the last part,

$$f_Y(y) = \int_{-\infty}^{\infty} 6(1-y)\chi_{0\le x\le y\le 1}\ \mathrm{d}x = \chi_{0\le y\le 1}\int_{0}^{y} 6(1-y)\ \mathrm{d}x = 6y(1-y)\chi_{0\le y\le 1}$$

So the conditional density of X given $Y = y$ is

$$f_{X|Y}(x|y) = f_{X,Y}(x,y)/f_Y(y) = \frac{6(1-y)\chi_{0\le x\le y\le 1}}{6y(1-y)\chi_{0\le y\le 1}} = \frac{1}{y}\chi_{0\le x\le y\le 1}$$

(c) To check if X and Y are independent, we compare the marginal density f_X and the conditional density $f_{X|Y}$ given $Y = y$ and check if they are equal on all x, y with $f_Y(y) > 0$.

By the previous parts, $f_X(x) = 3(1-x)^2\chi_{0\le x\le 1}$ and $f_{X|Y}(x|y) = \frac{1}{y}\chi_{0\le x\le y}$ which are two distinct functions, so X and Y are not independent.

(d)

$$\begin{aligned} P\left(Y \ge \frac{3}{4}\middle|X = \frac{1}{2}\right) &= \int_{-\infty}^{\infty} \chi_{y\ge\frac{3}{4}} f_{Y|X}(y|x = \frac{1}{2})\ \mathrm{d}y = \int_{-\infty}^{\infty} \chi_{y\ge\frac{3}{4}} \frac{f_{X,Y}(\frac{1}{2},y)}{f_X(\frac{1}{2})}\ \mathrm{d}y = \int_{\frac{3}{4}}^{\infty} \frac{6(1-y)\chi_{\frac{1}{2}\le y\le 1}}{\frac{3}{4}}\ \mathrm{d}y \\ &= \int_{\frac{3}{4}}^{1} 8(1-y)\ \mathrm{d}y = \frac{1}{4} \end{aligned}$$

(e) $\mathrm{E}\left(X - 3Y\right) = \mathrm{E}\left(X\right) - 3\mathrm{E}\left(Y\right) = \int_{-\infty}^{\infty} x f_X(x)\ \mathrm{d}x - 3\int_{-\infty}^{\infty} y f_Y(y)\ \mathrm{d}y = \int_0^1 x\cdot 3(1-x)^2\ \mathrm{d}x - 3\int_0^1 y \cdot 6y(1-y)\ \mathrm{d}y = \frac{1}{4} - 3\cdot\frac{1}{2} = -\frac{5}{4}$

Alternatively, $\mathrm{E}\left(X - 3Y\right) = \int_{0\le x\le y\le 1}(x-3y)6(1-y)\ \mathrm{d}x\ \mathrm{d}y = -\frac{5}{4}$

2. Consider two independent random variables X and Y. The pdf of X is given as

$$P(X = i) = \frac{1}{3} \quad \text{for } i = -1, 0, 1$$

and the pdf of Y is given as

$$f_Y(y) = \begin{cases} 1 & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Define $Z = X + Y$

(a) Compute $P\left(Z \le \frac{1}{2} \middle| X = 0\right)$

(b) Find the pdf of Z

Solution:

(a) $P\left(Z \le \frac{1}{2} \middle| X = 0\right) = P\left(X + Y \le \frac{1}{2} \middle| X = 0\right) = P\left(Y \le \frac{1}{2} \middle| X = 0\right) = P\left(Y \le \frac{1}{2}\right) = \int_{-\infty}^{\infty} \chi_{y \le \frac{1}{2}} f_Y(y)\, \mathrm{d}y = \frac{1}{2}$ due to the independence

(b) We first consider the cdf of Z. The cdf of Z is

$$\begin{aligned} F_Z(z) = P(Z \le z) &= P(X + Y \le z) \\ &= P(X + Y \le z | X = -1) P(X = -1) + P(X + Y \le z | X = 0) P(X = 0) \\ &\quad + P(X + Y \le z | X = 1) P(X = 1) \\ &= \frac{1}{3} \left(P(Y \le z + 1) + P(Y \le z) + P(Y \le z - 1)\right) = \frac{1}{3} \left(F_Y(z + 1) + F_Y(z) + F_Y(z - 1)\right) \end{aligned}$$

this means that the pdf of Z is

$$\begin{aligned} f_Z(z) &= \frac{1}{3} \left(f_Y(z + 1) + f_Y(z) + f_Y(z - 1)\right) = \frac{1}{3} \left(\chi_{0 \le z+1 \le 1} + \chi_{0 \le z \le 1} + \chi_{0 \le z-1 \le 1}\right) \\ &= \frac{1}{3} \chi_{-1 \le z \le 2} = \begin{cases} \frac{1}{3} & \text{if } -1 \le z \le 2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Note

Technically, the pdf from this computation is $f_Z(z) = \begin{cases} \frac{1}{3} & \text{if } z \in [-1, 0) \cup (0, 1) \cup (1, 2] \\ \frac{2}{3} & \text{if } z \in \{\, 0, 1 \,\} \\ 0 & \text{otherwise} \end{cases}$. However, pdf is only determined up to a set of measure zero, so in this view this is the same as the answer given above.