Chapter 3:
Markov Jump Process

§3.1 Introduction

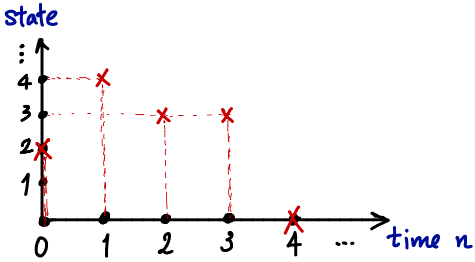
- **Jump process.**

Recall: a MC (**discrete-time** stochastic process with the Markovian property):

$$X(n) \in S, \quad n = 0, 1, 2, \cdots .$$

S: finite or countably infinite,

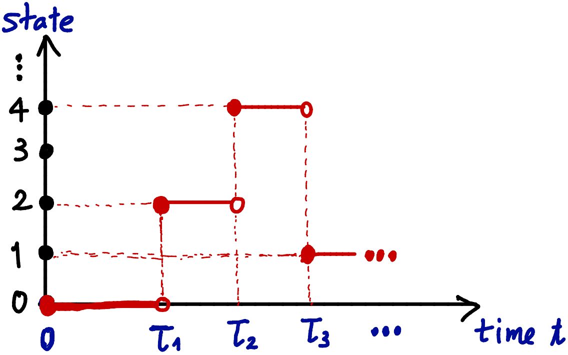
e.g. $S = \{0, 1, ..., N\}$ $(N \leq \infty)$.



Consider a **continuous-time** stochastic process:

$$X(t) \in S, \quad 0 \le t < \infty,$$

S: finite or countably infinite.



– $\tau_1, \tau_2, \cdots$: the waiting time to jump (random).

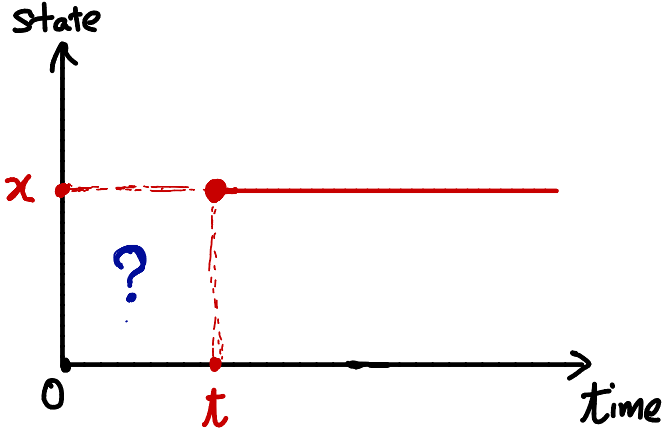
– $X(\tau_1), X(\tau_2), \cdots$: where to jump (random).

– Always assume: $\lim_{n\to\infty} \tau_n = \infty$ (No blow-up!)

- **Probability structure.**

Def.: $x \in S$ is **absorbing if**

"$X(t) = x$ for some $t \geqslant 0$" $\Rightarrow$ "$X(s) = x, \forall\, s \geqslant t$".



Given a **non-absorbing** state $X(0) = x \in S$, we need to know two things:

(i) $F_x(t)$, $t \geqslant 0$: the distribution of the waiting time τ_1. Note:

$$F_x(t) = P_x(\tau_1 \leq t).$$

(ii) Q_{xy}: the transition prob to jump from a state x to another state $y (\neq x)$:

$$Q_{xx} = 0, \quad \sum_{y \in S} Q_{xy} = 1.$$

(If x is **absorbing**, $Q_{xy} = \delta_{xy} = \begin{cases} 1, \text{for } x = y, \\ 0, \text{otherwise.} \end{cases}$)

For **non-absorbing** x, we assume:

$$P_x(\tau_1 \leqslant t, X(\tau_1) = y) = P_x(\tau_1 \leqslant t) Q_{xy},$$

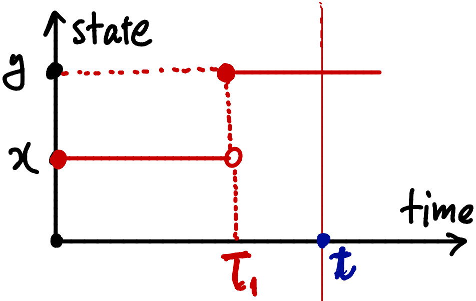
i.e.

τ_1 (**the waiting time to jump**)

and

$X(\tau_1)$ (**jump to where**)

are **independent**!



Similar to the MC (discrete-time), **our concern** is to determine the **transition function**:

$$\boxed{P_{xy}(t) \stackrel{\text{def}}{=} P(X(t) = y | X(0) = x) = P_x(X(t) = y),}$$

i.e., the prob that the process starting at x will be at y at time $t \geqslant 0$.

Note:

(i) $\sum_y P_{xy}(t) = 1$, $P_{xy}(0) = \delta_{xy}$.

(ii) If initial distribution is known, for instance, it is given by $\pi_0(x)$, $x \in S$, then

$$P(X(t) = y) = \sum_{x \in S} \pi_0(x) P_{xy}(t),$$

or $\pi_t = \pi_0 P(t)$ in matrix form.

- **Markov property:**

$$P(X(t) = y | X(s_1) = x_1, \cdots, X(s_n) = x_n, X(s) = x) \\ = P(X(t) = y | X(s) = x), \\ \forall\, 0 \leqslant s_1 \leqslant \cdots \leqslant s_n \leqslant s \leqslant t, \forall\, x_1, \cdots, x_n, x, y \in S.$$

Note:

- We always assume the process is **time-homogeneous**:

$$P(X(t) = y | X(s) = x) = P(X(t-s) = y | X(0) = x), \\ \forall\, 0 \leqslant s \leqslant t, \forall\, x, y \in S.$$

  Therefore

$$P(X(t) = y | X(s) = x) = P_{xy}(t-s).$$

- A Markov jump process (**MJP**) $\stackrel{\text{def}}{=}$ a continuous-time jump process with the Markovian property.

Now, we always consider the **MJP**.

Q.: How to determine $F_x(t) = P_x(\tau_1 \leq t)$?

Recall that $F_x(t)$ is the distribution of τ_1 (the waiting time for a jump to occur!).

To show: τ_1 is an **exponential** rv with density:

$$f(t) = \lambda e^{-\lambda t}, \quad t \geq 0; \quad \lambda \stackrel{\text{def}}{=} \frac{1}{E(\tau_1)}.$$

Hence:

$$F_x(t) = P(\tau_1 \leq t) = \int_{-\infty}^{t} f(s)\, ds = 1 - e^{-\lambda t},\ t \geq 0.$$

Def.: Let τ be a r.v. taking values in $[0, \infty)$. Then τ is said to be **memoryless** **if**

$$P(\tau > s + t | \tau > s) = P(\tau > t), \quad \forall\, s, t \geqslant 0,$$

(i.e., after waiting for time s, the prob for waiting for another time t has no memory that it already waits for time s.)

e.g. Model: Wait for an unreliable bus driver. Then, the waiting time is a memoryless r.v.:

> "If we have been waiting for s units of time then the prob we must wait t more units of time is the same as if we have not waited at all!"

Proposition. Let τ be a **memoryless** r.v. Then τ is an **exponential** r.v., and the density is given by

$$\lambda e^{-\lambda t},\ t \geqslant 0; \quad \lambda = 1/E(\tau).$$

Pf.: Let $G(t) \stackrel{\text{def}}{=} P(\tau > t)$. As τ is memoryless,

$$G(t) = P(\tau > t) = P(\tau > s + t | \tau > s) = \frac{P(\tau > s + t)}{P(\tau > s)} = \frac{G(s + t)}{G(s)},$$

i.e.

$$G(s + t) = G(s)G(t), \quad \forall\, s, t \geqslant 0.$$

Assuming G is differentiable,

$$\begin{aligned} G'(t) &= \lim_{h\to 0_+} \frac{G(t+h)-G(t)}{h} \\ &= \lim_{h\to 0_+} \frac{G(t)G(h)-G(t)}{h} \\ &= G(t)\lim_{h\to 0_+} \frac{G(h)-1}{h} \\ &\stackrel{\text{def}}{=} G(t)\alpha. \end{aligned}$$

Note: $G(0) = P(\tau > 0) = 1$. $\therefore$ $G(t) = e^{\alpha t}$.

Note: $G(t) = P(\tau > t)$ is decreasing. $\therefore \alpha < 0$. Set $\alpha = -\lambda$ ($\lambda > 0$). The density function is

$$f(t) = (1 - G(t))' = \lambda e^{-\lambda t}. \quad \square$$

Proposition. Let $X(t)$, $t \geqslant 0$ be a MJP. For a non-absorbing state $x \in S$, letting $X(0) = x$,

$$\tau_x \stackrel{\text{def}}{=} \inf\{t > 0 : X(t) \neq x\}. \quad \text{(first time to jump)}$$

Then, τ_x is a memoryless r.v.

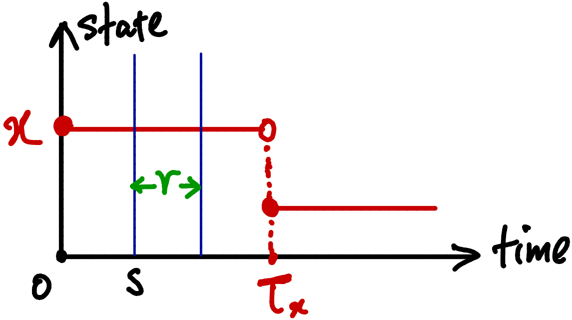
Pf.:

$$\begin{aligned}
&P(\tau_x > s + r | \tau_x > s) \\
&= P(X(t) = x, 0 \leqslant t \leqslant s + r | X(t) = x, 0 \leqslant t \leqslant s) \\
&= P(X(t) = x, s \leqslant t \leqslant s + r | X(t) = x, 0 \leqslant t \leqslant s) \\
&= P(X(t) = x, s \leqslant t \leqslant s + r | X(s) = x) \text{ (Markovian)} \\
&= P(X(t) = x, 0 \leqslant t \leqslant r | X(0) = x) \text{ (time-homog)} \\
&= P(\tau_x > r). \quad \square
\end{aligned}$$

Remarks:

- For a MJP, as τ_x is memoryless:

$$P(\tau_x > s + r | \tau_x > s) = P(\tau_x > r),$$



it looks like that the process starts from s.

- Set $q_x \stackrel{\text{def}}{=} 1/E(\tau_x)$. Then, τ_x has an exponential density given by $q_x e^{-q_x t}$ $(t \geq 0)$.

§3.2 Poisson process

We shall give the definition of **Poisson process** in terms of the **waiting time**.

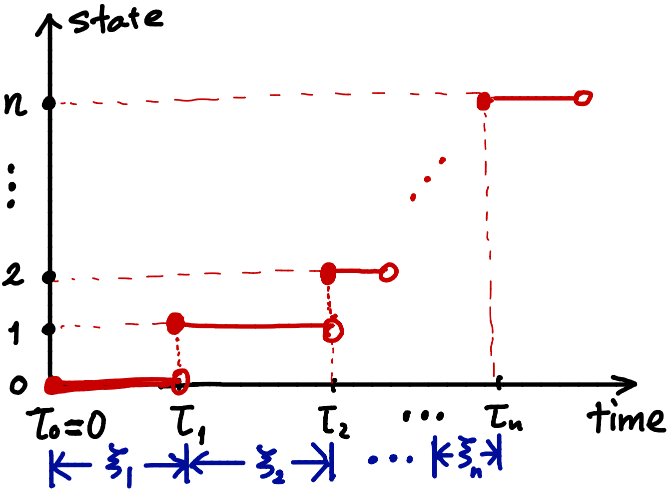
Setup:

• Let $\xi_n \sim \xi$, $n = 1, 2, \cdots$, be i.i.d. exp. r.v. with parameter λ:

$$P(\xi > t) = e^{-\lambda t}, \quad \lambda = 1/E(\xi).$$

• Define $\tau_0 = 0$, and

$$\tau_n \stackrel{\text{def}}{=} \xi_1 + \xi_2 + \cdots + \xi_n, \quad n = 1, 2, \cdots$$



For $n = 1, 2, \cdots,$

$\xi_n \sim \xi$: the waiting time for one arrival.

τ_n: the waiting time for the n^{th}-arrival.

For $t \geqslant 0$,

$$X(t) \stackrel{\text{def}}{=} \max\{n \geqslant 0, \tau_n \leqslant t\},$$

i.e., the no of arrival in $[0, t]$.

Then, we get a jump process:

$$X(t) \in \{0, 1, 2, \cdots\}, \quad t \geqslant 0.$$

Q.:

- **What's the density of $X(t)$? (Poisson with rate λt!)**
- **Is $X(t)$ a MJP? (YES!)**

Theorem. $X(t)$ is Poisson with $E(X(t)) = \lambda t$:

$$P(X(t) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n = 0, 1, 2 \cdots .$$

Pf.: By definition,

$$\{X(t) = n\} = \{\tau_n \leqslant t < \tau_{n+1}\} = \{\tau_{n+1} > t\} \backslash \{\tau_n > t\}.$$

Hence,

$$P(X(t) = n) = P(\tau_{n+1} > t) - P(\tau_n > t). \qquad (*)$$

- $n = 0$:

$$P(X(t) = 0) = P(\tau_1 > t) - 0 = P(\xi_1 > t) = e^{-\lambda t}.$$

- **To show:**

$$P(\tau_n > t) = e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}, \qquad (**)$$

$$n = 1, 2, \cdots .$$

If so, substituting $(**)$ into $(*)$ gives the theorem.

Proof of $(**)$ by induction:

$n = 1$: $P(\tau_1 > t) = P(\xi_1 > t) = e^{-\lambda t}$. $(**)$ holds.

Letting $(**)$ hold for $n \geq 1$, we need to show that $(**)$ is true for $n + 1$. Indeed,

$$
\begin{aligned}
&P(\tau_{n+1} > t) \\
&= P(\tau_n + \xi_{n+1} > t) \\
&= P(\xi_{n+1} > t) + P(\xi_{n+1} \le t, \tau_n + \xi_{n+1} > t) \\
&= e^{-\lambda t} + \int_0^t \lambda e^{-\lambda s} \cdot P(\tau_n > t - s)\, ds \text{ (explain later)} \\
&= e^{-\lambda t} + \int_0^t \lambda e^{-\lambda s} \cdot \sum_{k=0}^{n-1} e^{-\lambda(t-s)} \frac{(\lambda(t-s)^k)}{k!}\, ds
\end{aligned}
$$

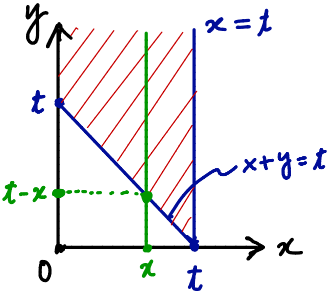
(Use induction assumption!)

$$
\begin{aligned}
&= e^{-\lambda t} + e^{-\lambda t} \sum_{k=0}^{n-1} \frac{\lambda^{k+1}}{k!} \cdot \int_0^t (t-s)^k ds \\
&= e^{-\lambda t} + e^{-\lambda t} \sum_{k=0}^{n-1} \frac{\lambda^{k+1}}{k!} \cdot \frac{t^{k+1}}{(k+1)} \\
&= e^{-\lambda t} + e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^{k+1}}{(k+1)!} = e^{-\lambda t} \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!}. \quad \square
\end{aligned}
$$

Note (See Durrett P93-103):

Let X, Y be independent with densities $f(\cdot)$, $g(\cdot)$ over $[0, \infty)$, resp. Then,

$$P(X < t, X + Y > t) = \int_0^t \int_{t-x}^{\infty} f(x)g(y)\, dydx$$
$$= \int_0^t f(x) \int_{t-x}^{\infty} g(y)\, dydx = \int_0^t f(x)P(Y > t - x)dx.$$



Remarks:

- Note:

$$P(X(0)=k)=\delta_{0k}=\begin{cases}1 & k=0,\\ 0 & \text{otherwise.}\end{cases}$$

$$\begin{aligned}\therefore\ P(X(t)=n) &= \sum_{k=0}^{\infty} P(X(0)=k)P(X(t)=n|X(0)=k)\\ &= \sum_{k=0}^{\infty}\delta_{0k}P_{kn}(t)\\ &= P_{0n}(t).\end{aligned}$$

$$\therefore\ P_{0n}(t)=e^{-\lambda t}\frac{(\lambda t)^n}{n!},\quad n=0,1,2\cdots.$$

- $E(X(t))=\lambda t$ is the **expected no of arrivals in $[0,t]$**. λ is the **arrival rate**.

Corollary. The **Poisson process** $\{X(t)\}_{t\geqslant 0}$ with rate λt satisfies:

(i) $X(0) = 0$.

(ii) For $0 < s < t$, $X(t) - X(s)$ has Poisson distribution with mean $\lambda(t-s)$, and is independent of $X(s)$.

(iii) For $0 \leqslant t_1 \leqslant \cdots \leqslant t_n$,

$$X(t_2) - X(t_1), \cdots, X(t_n) - X(t_{n-1})$$

are independent.

Also, $\{X(t)\}_{t\geqslant 0}$ satisfies the Markov property with

$$E(X(t)) = \lambda t, \quad Var(X(t)) = \lambda t.$$

Remark: Very often, (i)(ii)(iii) are also used as the definition of Poisson process!

IDEA of Proof:

- (i): Obvious.
- (ii): For $0 < s < t$,

$$
\begin{aligned}
& P(X(t) - X(s) = n) \\
&= \textstyle\sum_{m=0}^{\infty} P(X(s) = m, X(t) = n + m) \\
&= \textstyle\sum_{m=0}^{\infty} P(X(s) = m) P(X(t) = n + m | X(s) = m) \\
&= \textstyle\sum_{m=0}^{\infty} P(X(s) = m) P_{m,n+m}(t - s) \\
&= \textstyle\sum_{m=0}^{\infty} P(X(s) = m) P_{0,n}(t - s) \\
&= P_{0,n}(t - s) \\
&= e^{-\lambda(t-s)} \frac{[\lambda(t - s)]^n}{n!}.
\end{aligned}
$$

$X(t) - X(s)$ is independent of $X(s)$ means
$P(X(t)-X(s) = n, X(s) = m) = P(X(t)-X(s) = n)P(X(s) = m)$,
equivalently

$$P(X(t) - X(s) = n | X(s) = m) = P(X(t) - X(s) = n).$$

Indeed, note

$$\mathrm{LHS} = P(X(t) = m+n | X(s) = m) = P_{m,m+n}(t-s) = P_{0,n}(t-s).$$

- (iii): Omit the proof. Intuitively clear (See P94-95 in Durrent Chapter 3)

- For Markov property: Check

$$P(X(t) = y | X(t_1) = x_1, \cdots, X(t_n) = x_n, X(s) = x) = P(X(t) = y | X(s) = x)$$

for any $0 \leqslant t_1 < t_2 < \cdots < t_n < s \leqslant t$. □

Sum: We see that the **Poisson process**

$$X(t), \quad t \geq 0,$$

turns out to be a **MJP** (continuous-time JP with the Markov property) with $X(0) = 0$ and the **transition function**:

For any $t \geq 0$ and any $x, y \in S = \{0, 1, 2 \cdots\}$,

$$P_{xy}(t) = \begin{cases} 0 & \text{if } x > y, \\ \\ = P_{0,y-x}(t) = e^{-\lambda t} \dfrac{(\lambda t)^{y-x}}{(y-x)!} & \text{if } x \leq y. \end{cases}$$

Here, $\lambda > 0$ is the arrival rate. □

§3.3 Basic properties of MJP

Let $\{X(t)\}_{t\geqslant 0}$ be a MJP with

$$P_{xy}(t) = P(X(t) = y | X(0) = x).$$

Proposition. (Chapman-Kolmogorov equation)

$$P_{xy}(t+s) = \sum_z P_{xz}(t) P_{zy}(s).$$

In matrix form, letting $P(t) = [P_{xy}(t)]$, the above is

$$\boxed{P(t+s) = P(t)P(s).}$$

Remark: It is similar to the discrete case

$$P^{m+n}(x,y) = \sum_{z\in s} P^m(x,z)P^n(z,y)$$

Pf.: Note:

$$P_{xy}(t+s) = \sum_z P_x(X(t)=z, X(t+s)=y)$$

and

$$P_x(X(t)=z, X(t+s)=y)$$

$$= P_x(X(t)=z)P_x(X(t+s)=y|X(t)=z)$$

$$= P_x(X(t)=z)P(X(t+s)=y|X(0)=x, X(t)=z)$$

$$= P_x(X(t)=z)P(X(s)=y|X(0)=z) \text{ (Markov+Time-Homg)}$$

$$= P_{xz}(t)P_{zy}(s).$$

It follows that

$$P_{xy}(t+s) = \sum_z P_{xz}(t)P_{zy}(s). \quad \square$$

Note: Assume $P(t)$ is differentiable in $[0, \infty)$, and

$$\boxed{D \stackrel{\text{def}}{=} P'(0).}$$

Then, from the C.-K. equation

$$P(t+s) = P(t)P(s),$$

one has

$$\left.\frac{d}{ds}\right|_{s=0} (\cdot) \Rightarrow P'(t) = P(t)D,$$
$$\left.\frac{d}{dt}\right|_{t=0} (\cdot) \Rightarrow P'(s) = DP(s).$$

$$\therefore \boxed{P'(t) = P(t)D = DP(t), \quad t \geq 0.}$$

Fact I.

$$D = P'(0) \stackrel{\text{def}}{=} [q_{xy}]_{x,y \in S} = \begin{bmatrix} - & + & + & + & \cdots \\ + & - & + & + & \cdots \\ + & + & - & + & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

called the **rate matrix**.

$$+ : \text{entry} \geq 0; \quad - : \text{entry} \leq 0.$$

Indeed, note:

$$
\begin{aligned}
q_{xy} &= P'_{xy}(0) \\
&= \lim_{h\to 0+} \frac{P_{xy}(h) - P_{xy}(0)}{h} \\
&= \lim_{h\to 0+} \frac{P(X(h) = y | X(0) = x) - P(X(0) = y | X(0) = x)}{h} \\
&= \begin{cases} \lim_{h\to 0+} \dfrac{P(X(h) = y | X(0) = x) - 1}{h} (\leqslant 0) & \text{if } x = y, \\ \lim_{h\to 0+} \dfrac{P(X(h) = y | X(0) = x) - 0}{h} (\geqslant 0) & \text{if } x \neq y. \end{cases}
\end{aligned}
$$

Fact II. Each row sum of D is zero:

$$\sum_{y \in S} q_{xy} = 0, \quad \forall\, x \in S. \qquad (*)$$

Indeed, note:

$$\sum_{y \in S} P_{xy}(t) = 1, \forall\, t \geq 0. \;\; \therefore\; \frac{d}{dt}\Big|_{t=0} \Rightarrow \sum_{y \in S} P'_{xy}(0) = 0. \qquad \square$$

Observe: $(*)$ means $q_{xx} + \sum_{y \neq x} q_{xy} = 0$, that is,

$$\boxed{\underbrace{-q_{xx}}_{\text{the rate to jump away from } x} = \sum_{y \neq x} \underbrace{q_{xy}}_{\text{the rate to jump to } y \text{ from } x}.}$$

Recall:

- $E(\tau_x)$ is the mean waiting time to jump away from x, so $q_x = \frac{1}{E(\tau_x)}$ is **the rate of change. Note:**

$$q_x = 0 \textbf{ iff } E(\tau_x) = \infty, \textbf{ iff } x \text{ is absorbing.}$$

- $Q = [Q_{xy}]$ is the Markov matrix introduced before. $Q_{xx} = 1$ **iff** x is absorbing. For non-absorbing x,

$$Q_{xx} = 0, \quad \sum_{y \neq x} Q_{xy} = 1,$$

and in such case, Q_{xy} is understood to be **the proportion that the chain will jump to y from x**.

Main Theorem:

$$-q_{xx} = q_x; \quad q_{xy} = q_x Q_{xy} \text{ for } y \neq x.$$

Pf.: **Case x is absorbing** ($q_x = 0, Q_{xy} = \delta_{xy}$):

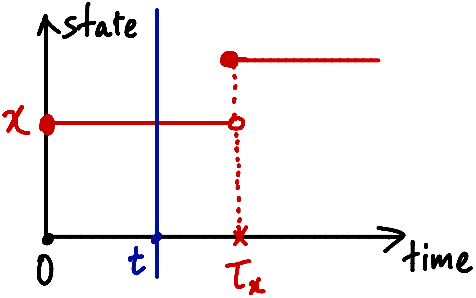
$$P_{xy}(t) = \delta_{xy}. \quad \therefore\ q_{xy} = P'_{xy}(0) = 0.$$

Conclusion is then TRUE.

Case x is non-absorbing:

$$\begin{aligned} P_{xy}(t) &= P_x(X(t) = y) \\ &= \underbrace{P_x(\tau_x > t, X(t) = y)}_{I:\ \text{no jump yet}} \\ &\quad + \underbrace{P_x(\tau_x \leqslant t, X(t) = y)}_{II:\ \text{it has jumped}}. \end{aligned}$$

For I (no jump yet):

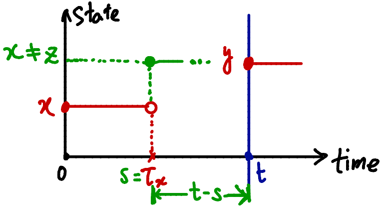


$$I = P_x(\tau_x > t, X(t) = y)$$

$$= \begin{cases} 0 & \text{for } y \neq x, \\ P_x(\tau_x > t) = e^{-q_x t} & \text{for } y = x \end{cases}$$

$$= \delta_{xy} e^{-q_x t}.$$

For *II* (it has jumped):



$$II = P_x(\tau_x \leqslant t, X(t) = y)$$
$$= \sum_{z \neq x} P_x(\tau_x \leqslant t, X(\tau_x) = z, X(t) = y)$$
$$= \sum_{z \neq x} \int_0^t P_x(\tau_x = s) Q_{xz} P_{zy}(t-s) ds$$
$$= \sum_{z \neq x} \int_0^t q_x e^{-q_x s} Q_{xz} P_{zy}(t-s) ds.$$

$$\therefore P_{xy}(t) = I + II$$

$$= \delta_{xy} e^{-q_x t} + \sum_{z \neq x} \int_0^t q_x e^{-q_x s} Q_{xz} P_{zy}(t-s) ds$$

$$= \delta_{xy} e^{-q_x t} + q_x e^{-q_x t} \sum_{z \neq x} \int_0^t Q_{xz} P_{zy}(u) e^{q_x u} du$$

(Change of variable: $t - s = u$)

$$\therefore P'_{xy}(t) = -q_x P_{xy}(t) + q_x \sum_{z \neq x} Q_{xz} P_{zy}(t)$$

$$\therefore P'_{xy}(0) = -q_x P_{xy}(0) + q_x \sum_{z \neq x} Q_{xz} P_{zy}(0)$$

$$= -q_x \delta_{xy} + q_z \sum_{z \neq x} Q_{xz} \delta_{zy}$$

$$= -q_x \delta_{xy} + q_x Q_{xy}$$

$$= \begin{cases} -q_x + 0 = -q_x & \text{for } y = x, \\ q_x Q_{xy} & \text{for } y \neq x. \end{cases} \qquad \square$$

Example 1. Poisson process with rate λt:

$$P_{0n}(t) = P(X(t) = n | X(0) = 0) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \cdots$$

$$P(t) = \begin{bmatrix} e^{-\lambda t} & e^{-\lambda t}\frac{\lambda t}{1!} & e^{-\lambda t}\frac{(\lambda t)^2}{2!} & \cdots \\ 0 & e^{-\lambda t} & e^{-\lambda t}\frac{\lambda t}{1!} & \cdots \\ 0 & 0 & \ddots & \ddots \\ 0 & 0 & 0 & \ddots \end{bmatrix}$$ **(transition function)**

Then

$$D = P'(0) = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots & \cdots \\ 0 & -\lambda & \lambda & 0 & \cdots \\ 0 & 0 & \ddots & \ddots & \ddots \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & \cdots \\ 0 & 0 & \ddots & \ddots & \ddots \end{bmatrix}.$$

Example 2. Car check up with 3 operations in sequence:
(1) Engine time up $\to$ (2) air condition repair $\to$ (3) break system replacement $\to$ (4) leave.

Assume that this is a MJP with the mean time in each operation 1.2, 1.5, 2.5 hours.

$S = \{1, 2, 3, 4\}$. The rate of moving up to the next stage is $\frac{1}{1.2}$, $\frac{1}{1.5}$, $\frac{1}{2.5}$. Thus,

$$D = \begin{bmatrix} -\frac{1}{1.2} & \frac{1}{1.2} & 0 & 0 \\ 0 & -\frac{1}{1.5} & \frac{1}{1.5} & 0 \\ 0 & 0 & -\frac{1}{2.5} & \frac{1}{2.5} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Further questions:

(a) What is the prob that after 4hour the car is in step (3)? That is to find $P(X(4) = 3|X(0) = 1)$.

(b) What is the prob that after 4hour the car is still in the shop? That is to find $P(X(4) = 4|X(0) = 1)$.

Generally, need to find

$$P(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) & P_{13}(t) & P_{14}(t) \\ P_{21}(t) & P_{22}(t) & P_{23}(t) & P_{24}(t) \\ P_{31}(t) & P_{32}(t) & P_{33}(t) & P_{34}(t) \\ P_{41}(t) & P_{42}(t) & P_{43}(t) & P_{44}(t) \end{bmatrix}.$$

Method: Solve the linear ODE system:

$$P'(t) = DP(t), \quad P(0) = I.$$

Example 3. A barbar shop with two barbars and two waiting chains. Customers arrives at a rate 5 per hr. Each barbar serves at a rate 2 per hr. If the waiting chains are full the customer will leave.

$$X(t) \stackrel{\text{def}}{=} \text{the no of customers in the shop.}$$
$$S = \{0, 1, 2, 3, 4\}.$$

$$D = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -5 & 5 & 0 & 0 & 0 \\ 2 & -7 & 5 & 0 & 0 \\ 0 & 4 & -9 & 5 & 0 \\ 0 & 0 & 4 & -9 & 5 \\ 0 & 0 & 0 & 4 & -4 \end{bmatrix} \end{array}, \quad Q = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{2}{7} & 0 & \frac{5}{7} & 0 & 0 \\ 0 & \frac{4}{9} & 0 & \frac{5}{9} & 0 \\ 0 & 0 & \frac{4}{9} & 0 & \frac{5}{9} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Further questions:

(a) In the long run, what is the prob to have one customer, two customers, etc.? That is to find

$$\lim_{t\to\infty} P(X(t) = k), \quad k \in S.$$

(b) Find the expected time for it to be full, counting from the opening time. That is to find

$$E(T_y),$$

where $T_y = \inf\{t : X(t) = y, X(0) = 0\}$.

How to solve:

$$P'(t) = DP(t), \quad P(0) = I.$$

Case when S **is finite**:

$$P(t) = e^{tD} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(tD)^n}{n!} \quad \text{(convergent!)}.$$

Informal Proof: At $t = 0$, $e^{tD} = e^{0D} = I$ (Convention: $0^0 = 1$, $D^0 = I$), and for $t > 0$,

$$\begin{aligned}
(e^{tD})' &= \sum_{n=1}^{\infty} \frac{t^{n-1} D^n}{(n-1)!} \\
&= D \left[\sum_{n=1}^{\infty} \frac{(tD)^{n-1}}{(n-1)!} \right] \\
&= De^{tD}. \quad \square
\end{aligned}$$

Example. Let $D = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$. **Q.:** Find $P(t)$.

Sol.: Look for $D = Q \operatorname{diag}\{\lambda_1, \lambda_2\} Q^{-1}$.

(i) **Eigenvalues:** $\det(D - \lambda I) = 0$,
i.e., $0 = \det \begin{bmatrix} -1-\lambda & 1 \\ 2 & -2-\lambda \end{bmatrix} = (-1-\lambda)(-2-\lambda) - 2$,
i.e., $\lambda^2 + 3\lambda = 0$. $\therefore \lambda = 0, -3$.

(ii) **Eigenvectors:** $\lambda = 0 : D - \lambda I = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}, e_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\lambda = -3 : D - \lambda I = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}, e_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Let $Q \overset{\text{def}}{=} [e_1, e_2] = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$, $Q^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$. Then,

$$\begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} = Q^{-1} D Q, \quad i.e., \boxed{D = Q \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} Q^{-1}.}$$

Hence

$$P(t) = e^{tD} = \sum_{n=0}^{\infty} \frac{(tD)^n}{n!} = Q \left(\sum_{n=0}^{\infty} \frac{\left(t \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \right)^n}{n!} \right) Q^{-1}$$

$$= Q \begin{bmatrix} \sum_{n=0}^{\infty} \frac{0^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{(-3t)^n}{n!} \end{bmatrix} Q^{-1}$$

$$= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} + e^{-3t} \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix},$$

$$\therefore \quad \lim_{t\to\infty} P(t) = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \quad \text{namely,}$$

$$\lim_{t\to\infty} P(X(t) = 0) = 2/3, \ \lim_{t\to\infty} P(X(t) = 1) = 1/3. \quad \square$$

Remark: Set $\pi = [2/3, 1/3]$. Then, $\pi P(t) = \pi$, $\forall\, t \geq 0$, so π is a **SD** for $P(t)$.

§3.4 The birth and death process

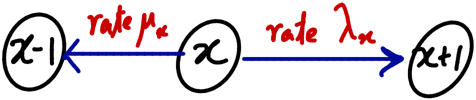
Setup:

Let $S = \{0, 1, \cdots\}$,

$$D = [q_{xy}] = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots \end{bmatrix}.$$

Assume that all $\lambda_x, \mu_x \neq 0 \, (> 0)$.

λ_x: birth rate, μ_x: death rate



Example 1. Revisit the Poisson process.

We already derived earlier $P(t) = [P_{xy}(t)]$ for a Poisson process $X(t)$, $t \geqslant 0$, using

$$X(t) = \max\{n : \tau_n \leqslant t\}.$$

We further have derived:

$$P'(t) = P(t)D, \quad D = \begin{bmatrix} -\lambda & \lambda & & \\ & -\lambda & \lambda & \\ & & \ddots & \ddots \end{bmatrix},$$

$\lambda > 0$: arrival rate.

Here we want to derive the **inverse**:

Proposition. If $X(t)$ is a MJP with rate matrix

$$D = \begin{bmatrix} -\lambda & \lambda & & \\ & -\lambda & \lambda & \\ & & \ddots & \ddots \end{bmatrix},$$

then $X(t)$ has the Poisson distribution, i.e.

$$P_{xy}(t) = \begin{cases} e^{-\lambda t} \frac{(\lambda t)^{(y-x)}}{(y-x)!} & \text{if } y \geqslant x \geqslant 0, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

It is **another way** of obtaining the Poisson process.

Pf.: Recall

$$P'_{xy}(t) = \sum_z P_{xz}(t) q_{zy}.$$

Observe

(i) If $y = 0$, then

$$P'_{x0}(t) = -\lambda P_{x0}(t), \quad P_{x0}(0) = \delta_{x0}.$$

$$\therefore P_{x0}(t) = \delta_{x0} e^{-\lambda t}.$$

(ii) If $y \geq 1$, then

$$P'_{xy}(t) = \lambda P_{x,y-1}(t) - \lambda P_{xy}(t), \quad P_{xy}(0) = \delta_{xy}.$$

$$\therefore P_{xy}(t) = e^{-\lambda t}\delta_{xy} + \int_0^t e^{-\lambda(t-s)} \lambda P_{x,y-1}(s)\, ds.$$

Claim #1. $P_{xy}(t) = 0, \forall\, y < x$. Indeed,

if $y = 0$ $(x \geq 1)$, $P_{x0}(t) = 0$.

if $y = 1$ $(x \geq 2)$,

$$P'_{x,1}(t) = \lambda P_{x,0}(t) - \lambda P_{x,1}(t) = -\lambda P_{x,1}(t), \;\; P_{x,1}(0) = 0.$$

$$\therefore P_{x,1}(t) = 0.$$

If $y = 2$ $(x \geq 3)$,

$$P'_{x,2}(t) = \lambda P_{x,1}(t) - \lambda P_{x,2}(t) = -\lambda P_{x,2}(t), \;\; P_{x,2}(0) = 0.$$

$$\therefore P_{x,2}(t) = 0.$$

Inductively,

$$P_{xy}(t) = 0, \quad \forall\, x > y \geq 0.$$

Claim #2. $P_{xy}(t) = e^{-\lambda t}\frac{(\lambda t)^{y-x}}{(y-x)!}$, $\forall\, y \geq x \geq 0$.

Indeed, let $x \geq 0$ be fixed.

For $y = x$,

$$P_{xx}(t) = e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)}\lambda \underbrace{P_{x,x-1}(s)}_{=0}\, ds = e^{-\lambda t}.$$

For $y = x + 1$,

$$P_{x,x+1}(t) = e^{-\lambda t}\underbrace{\delta_{x,x+1}}_{=0} + \int_0^t e^{-\lambda(t-s)}\underbrace{\lambda P_{x,x}(s)}_{=e^{-\lambda s}}\, ds$$

$$= \cdots = e^{-\lambda t}\lambda t.$$

Inductively, we get the desired result. □

Exercise:

(1) Extend the above to

$$D = \begin{bmatrix} -\lambda_0 & \lambda_0 & & & \\ & -\lambda_1 & \lambda_1 & & \\ & & -\lambda_2 & \lambda_2 & \\ & & & \ddots & \ddots \end{bmatrix}, \quad \text{(see. P.98).}$$

It is a general **pure birth** process.

(2) Think about the more general **BD process**:

$$D = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots \end{bmatrix}. \quad \square$$

Example 2. Branching Process:

– A collection of particles

– each waiting to
either split into two particles with prob p
or vanish with prob $(1-p)$

– the waiting time is exp. r.v. with rate λ.

$$X(t) \stackrel{\text{def}}{=} \text{be the no of particles at time } t.$$

Q.: Find the rate matrix D.

Lemma. Let $\xi_1, \cdots, \xi_n$ be independent r.v. having exponential distribution with rate $\alpha_1, \cdots, \alpha_n$, resp. Then,

$$\min\{\xi_1, \cdots, \xi_n\}$$

is an exponential r.v. with rate

$$\alpha_1 + \cdots + \alpha_n,$$

and for each $k = 1, \cdots, n$

$$P(\xi_k = \min\{\xi_1, \cdots, \xi_n\}) = \frac{\alpha_k}{\alpha_1 + \cdots + \alpha_n}. \quad \square$$

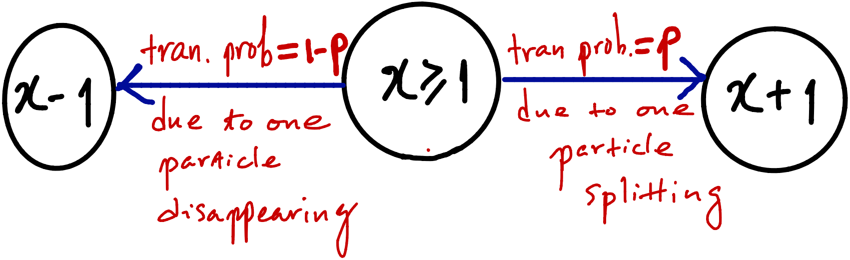
If so, then

$$Q = \begin{bmatrix} 1 & 0 & \cdots & & \\ 1-p & 0 & p & & \\ & 1-p & 0 & p & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \text{ (Markov matrix for state transition)},$$

$$D = \begin{bmatrix} 0 & 0 & 0 & & \\ (1-p)\lambda & -\lambda & p\lambda & & \\ & 2\lambda(1-p) & -2\lambda & 2\lambda p & \\ & & 3\lambda(1-p) & -3\lambda & 3\lambda p \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \text{ (rate matrix)}.$$

Indeed,

- Let $X(0) = x$, and $\xi_1, \cdots, \xi_x$ be **the time** any one of the particles splits or disappears.
- At time $\tau_1 = \min\{\xi_1, \cdots, \xi_x\}$, the no of particles will be $x + 1$ or $x - 1$.
- By lemma above, τ_1 is an exp. r.v. with rate λx:
  the portion to $x + 1 = p \cdot \lambda x$; the portion to $x - 1 = (1 - p) \cdot \lambda x$.



$$\lambda x = \boxed{\text{the rate to jump away from } x}$$

$$p \cdot \lambda x = \boxed{\text{the rate to jump to } x+1}$$

(Birth rate)

$$(1-p) \cdot \lambda x = \boxed{\text{the rate to jump to } x-1}$$

(Death rate)

Proof of Lemma:

$$
\begin{aligned}
& P(\min\{\xi_1, \cdots, \xi_n\} > t) \\
& = P(\xi_1 > t, \cdots, \xi_n > t) \\
& = P(\xi_1 > t) \times \cdots \times P(\xi_n > t) \\
& = e^{-\alpha_1 t} \times \cdots \times e^{-\alpha_n t} \\
& = e^{-(\alpha_1 + \cdots + \alpha_n) t}.
\end{aligned}
$$

To consider $P(\xi_k = \min\{\xi_1, \cdots, \xi_n\})$, W.L.G. take $k = 1$. Set

$$\eta = \min\{\xi_2, \cdots, \xi_n\}.$$

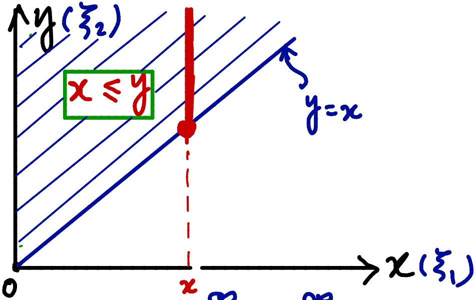
Then by above, η is an exp.r.v. with rate

$$\beta_1 \stackrel{\text{def}}{=} \sum_{y=2}^{n} \alpha_y.$$

$$
\begin{aligned}
& P(\xi_1 = \min\{\xi_1, \cdots, \xi_n\}) \\
&= P(\xi_1 \leqslant \eta) \\
&= \iint\limits_{x \leqslant y} \alpha_1 e^{-\alpha_1 x} \cdot \beta_1 e^{-\beta_1 y} dxdy \\
&= \int_0^\infty \left(\int_x^\infty \cdots dy \right) dx \\
&= \frac{\alpha_1}{\alpha_1 + \beta_1} \\
&= \frac{\alpha_1}{\alpha_1 + \sum_{y=2}^n \alpha_y} \\
&= \frac{\alpha_1}{\alpha_1 + \alpha_2 + \cdots + \alpha_n}. \quad \square
\end{aligned}
$$

For instance, consider ξ_1, ξ_2 only:

$$
\begin{aligned}
P(\xi_1 = \min\{\xi_1, \xi_2\}) &= P(\xi_1 \leqslant \xi_2) \\
&= \iint_{x \leqslant y} \alpha_1 e^{-\alpha_1 x} \cdot \alpha_2 e^{-\alpha_2 y} \, dx dy \\
&= \cdots = \frac{\alpha_1}{\alpha_1 + \alpha_2}.
\end{aligned}
$$



$$\iint_{x \leq y} = \int_0^\infty dx \int_x^\infty dy \ \cdots$$

Remark: Suppose that we allow **new particles** to immigrate into the system at **rate** α, and then give succeeding generation.

$$\eta \stackrel{\text{def}}{=} \textbf{the first time} \text{ a new particle arrives.}$$

$$\tau_1 = \min\{\xi_1, \cdots, \xi_x, \eta\}: \text{ the waiting time to change.}$$

$$\text{the rate of changing } \textbf{away from } x \text{ particles} = x\lambda + \alpha.\text{v}$$

$$D = \begin{bmatrix} -\alpha & \alpha & & & \\ (1-p)\lambda & -(\lambda+\alpha) & p\lambda+\alpha & & \\ & 2(1-p)\lambda & -(2\lambda+\alpha) & 2p\lambda+\alpha & \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix}$$

See the textbook P92. □

Example 3. Queuing Model.

$X(t) \stackrel{\text{def}}{=}$ the no of persons on the line at time t waiting for service.

$$\begin{cases} \text{arrival rate } \lambda : \text{Poisson} \\ \text{service rate } \mu\text{: exponential distr} \end{cases}$$

There are several models for queueing.

• **M/M/1 queue:**

M stands for memoyless,

1^{st} M stands for waiting time for the arrival,

2^{nd} M stands for waiting time for service,

The last number is for the number of servers.

$$D = \begin{bmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & (-\lambda + \mu) & \lambda & 0 \\ 0 & \ddots & \ddots & \ddots \end{bmatrix}.$$

• **M/M/k queue** (k servers):

Note: $\mu_n = \begin{cases} n\mu & \text{if } n \leqslant k, \\ k\mu & \text{if } n > k. \end{cases}$

$$D = \begin{bmatrix} -\lambda & \lambda & & & & & & \\ \mu & -(\mu+\lambda) & \lambda & & & & \mathbf{0} & \\ & 2\mu & -(2\mu+\lambda) & \lambda & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & k\mu & -(k\mu+\lambda) & \lambda & & \\ & \mathbf{0} & & & k\mu & -(k\mu+\lambda) & \lambda & \\ & & & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

- **M/M/∞ queue** (∞ servers):

$$D = \begin{bmatrix} -\lambda & \lambda & & & \\ \mu & -(\mu+\lambda) & \lambda & & \\ & 2\mu & -(2\mu+\lambda) & \lambda & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

arrival rate $= \lambda$, service rate $= \mu$,

$X(t) \stackrel{\text{def}}{=}$ the no of customers on the line at time t.

(e.g., in the telephone exchange, this is a continuous-time version of a previous example in the Markov chain).

Q.: Find $P_{xy}(t)$ and $\lim_{t\to\infty} P_{xy}(t)$.

Lemma. Let $Y(t)$ be a Poisson process with rate λ. Then for $0 \leqslant s \leqslant t$ (t fixed),

$$P(\tau_1 \leqslant s | Y(t) = 1) = \frac{s}{t},$$

i.e. the density function is $\frac{1}{t}$ on $[0, t]$, namely, given that the arrival (one) is within $[0, t]$, the arrival time is a uniform distr on $[0, t]$.

Note: This is a special case of Ex 6 with $Y(t) = n$.

Pf.: For $0 \leqslant s \leqslant t$,

$$
\begin{aligned}
& P(\tau_1 \leqslant s | Y(t) = 1) \\
&= P(Y(s) = 1 | Y(t) = 1) \\
&= \frac{P(Y(s) = 1, Y(t) = 1)}{P(Y(t) = 1)} \\
&= \frac{P(Y(s) = 1, Y(t) - Y(s) = 0)}{P(Y(t) = 1)} \\
&= \frac{e^{-\lambda s} \frac{(\lambda s)}{1!} \cdot e^{-\lambda(t-s)} \frac{(\lambda(t-s))^0}{0!}}{e^{-\lambda t} \frac{(\lambda t)}{1!}} \\
&= \frac{s}{t}. \quad \square
\end{aligned}
$$

Assume $X(0) = x$.

$$Y(t) \stackrel{\text{def}}{=} \text{the total no that arrived in time } (0, t].$$

Let

$$X(t) = R(t) + N(t),$$

$$R(t) \stackrel{\text{def}}{=} \text{the no of the original } x \text{ (at } t = 0\text{) that are still being served,}$$

$$N(t) \stackrel{\text{def}}{=} \text{the no of those from } Y(t) \text{ that are still being served.}$$

Fact 1. $R(t)$, i.e., the no of the original x (at $t = 0$) that are still being served,

is a binomial r.v.:

$$P(R(t) = k) = \binom{x}{k} (e^{-\mu t})^k (1 - e^{-\mu t})^{x-k},$$

$$0 \leq k \leq x,$$

$$x = \text{the total no at } t = 0,$$

$e^{-\mu t}$ = **the success prob** of still being served.

Fact 2. Recall: $Y(t)$ is the total no that arrived in time $(0, t]$. We want to consider

$$P(N(t) = n | Y(t) = k).$$

Note: Fix t.

- Given $Y(t) = k$, $N(t)$ should be a binomial r.v., but we have to find "**the success prob**":

$$p_t = P(N(t) = 1 | Y(t) = 1).$$

- For one that arrived at time $s \in (0, t]$, the prob of still being served at time t is $e^{-\mu(t-s)}$.
- By lemma, the arrival time s subject to one arrival in $(0, t]$ is uniform dist $1/t$.
- Then the prob that he is still being served at time t is

$$p_t = \int_0^t \frac{1}{t} \cdot e^{-\mu(t-s)} ds = \frac{1 - e^{-\mu t}}{\mu t}.$$

Hence,

$$P(N(t) = n | Y(t) = k) = \binom{k}{n} p_t^n (1 - p_t)^{k-n},$$

$$0 \leq n \leq k.$$

$$\begin{aligned} \therefore\ P(N(t) = n) &= \sum_{k=n}^{\infty} P(Y(t) = k, N(t) = n) \\ &= \sum_{k=n}^{\infty} P(Y(t) = k) P(N(t) = n | Y(t) = k) \\ &= \cdots \\ &= \frac{(\lambda t p_t)^n}{n!} e^{-\lambda t p_t}. \quad \text{(see P101)} \end{aligned}$$

The same as in last Chap (P55). □

We conclude that (Recall $X(t) = R(t) + N(t)$)

$$\begin{aligned} P_{xy}(t) &= P_x(X(t) = y) \\ &= \sum_{k=0}^{\min\{x,y\}} P_x(R(t) = k)P(N(t) = y - k) \\ &= \sum_{k=0}^{\min\{x,y\}} \binom{x}{k} e^{-k\mu t}(1 - e^{-\mu t})^{x-k} \frac{(\lambda t P_t)^{y-k} e^{-\lambda t p_t}}{(y-k)!}. \end{aligned}$$

For $t \to \infty$, all the terms vanish except $k = 0$:

$$\lim_{t\to\infty} P_{xy}(t) = e^{-\lambda/\mu} \frac{(\lambda/\mu)^y}{y!} \quad (tp_t \to 1/\mu \quad \text{as } t \to \infty).$$

Note: Compare it with the "telephone exchange" example last chapter.

§3.5 Limiting properties of MJP

The definitions of

- stationary distribution (SD)
- recurrence or transience
- etc

are the same as Markov chain.

Let us only sketch some of them.

- **SD:**

Let

$$X(t), \quad t \geqslant 0,$$

be a MJP.

Def.: π is called a **SD if**

(i) (distribution)

$$\pi(y) \geqslant 0, \forall\, y \in S; \sum_{y} \pi(y) = 1.$$

(ii) (stationary)

$$\sum_{x \in S} \pi(x) P_{xy}(t) = \pi(y), \forall\, y \in S, \forall\, t \geqslant 0.$$

How to find the SD π?

In fact,

$$0 = \left(\sum_x \pi(x) P_{xy}(t) \right)' = \sum_x \pi(x) P'_{xy}(t).$$

(**Note:** there is a technical point to interchange $\sum_x$ and $(\cdot)'$ for the **infinite** sum)

Let $t \to 0+$, then $\sum_x \pi(x) q_{xy} = 0$, i.e. in matrix form

$$\pi D = 0,$$

where $D = [q_{xy}]$ is the rate matrix. The converse is also true.

Example. Find the SD of the birth and death process with rate

$$D = \begin{bmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Sol.: Let $\pi = (x_0, x_1, \cdots)$. $\pi D = 0$ is

$$[x_0, x_1, \cdots] \begin{bmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -(\mu_1+\lambda_1) & \lambda_1 & & \\ & & \ddots & \ddots & \ddots \end{bmatrix} = [0, 0, \cdots].$$

Hence

$$\begin{cases} -\lambda_0 x_0 + \mu_1 x_1 = 0, \\ \lambda_{k-1}x_{k-1} - (\lambda_k + \mu_k)x_k + \mu_{k+1}x_{k+1} = 0, \ \ k \geq 1. \end{cases}$$

Note: For $k \geq 1$,

$$\begin{aligned} \lambda_k x_k - \mu_{k+1}x_{k+1} &= \lambda_{k-1}x_{k-1} - \mu_k x_k \\ &= \cdots = \lambda_0 x_0 - \mu_1 x_1 = 0. \end{aligned}$$

$$\therefore x_k = \frac{\lambda_{k-1}}{\mu_k} x_{k-1} = \cdots = \frac{\lambda_{k-1}}{\mu_k} \cdot \frac{\lambda_{k-2}}{\mu_{k-1}} \cdots \frac{\lambda_0}{\mu_1} x_0,$$

$$\therefore x_k = \beta_k x_0, \quad \beta_k \stackrel{\text{def}}{=} \frac{\lambda_0 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k} \, (k \geq 1).$$

Formally, $\sum_{k=0}^{\infty} x_k = (\sum_{k=0}^{\infty} \beta_k) x_0$ (Convention: $\beta_0 = 1$).
Then,

- if $\beta \overset{\text{def}}{=} \sum_{k=0}^{\infty} \beta_k < \infty$, then choosing $x_0 = \frac{1}{\beta}$,

$$\pi = \left(\frac{1}{\beta}, \frac{\beta_1}{\beta}, \frac{\beta_2}{\beta}, \cdots \right) \text{ is a SD.}$$

- if $\sum_{k=0}^{\infty} \beta_k = \infty$, then, no SD! $\square$

Exercise: Use this to check the queue models:

M/M/1, M/M/2, M/M/∞.

For instance, M/M/∞ case:

$$\begin{cases} \lambda_k = \lambda \ (k \geqslant 0) \\ \mu_k = k\mu \ (k \geqslant 1) \end{cases} \quad \therefore \beta_k = \left(\frac{\lambda_0}{\mu_1}\right) \cdots \left(\frac{\lambda_{k-1}}{\mu_k}\right) = \frac{\lambda^k}{k!\mu^k}.$$

$$\sum_{k \geq 0} \beta_k = e^{\lambda/\mu}.$$

$$\therefore \pi = \left(e^{-\frac{\lambda}{\mu}}, e^{-\frac{\lambda}{\mu}} \frac{\lambda}{\mu}, \frac{e^{-\frac{\lambda}{\mu}} (\frac{\lambda}{\mu})^2}{2!}, \cdots, \frac{e^{-\frac{\lambda}{\mu}} (\frac{\lambda}{\mu})^k}{k!}, \cdots \right).$$

"the same as the one by looking for the limit distribution $\lim_{t \to \infty} P_{xy}(t)$"

- **Recurrence and transience.**

$\tau_1 \overset{\text{def}}{=}$ the first time to jump

$$T_y \overset{\text{def}}{=} \min\{t \geqslant \tau_1 : X(t) = y\} \text{ (hitting time)}$$

$$(= \infty \text{ if } X(t) \neq y, \forall\, t \geqslant \tau_1)$$

$$\rho_{xy} \overset{\text{def}}{=} P_x(T_y < \infty)$$

(the prob that the process starting from x eventually hits y)

Recurrent: $\rho_{yy} = 1$.
Transient: $\rho_{yy} < 1$.
Process is irreducible: $\rho_{xy} > 0$, $\forall\, x, y \in S$.

Let Q be the matrix in the MJP, i.e.

$$P_x(\tau_1 \leqslant t, X(\tau_1) = y) = F_x(t) Q_{xy},\ \ y \neq x,$$

$$F_x(t) = 1 - e^{-q_x t}.$$

Assume irreducible, i.e. $q_x > 0$, $\forall\, x$. Then

$$P(X(\tau_1) = y | X(0) = x) = Q_{xy} (= \frac{q_{xy}}{q_x}),\ \forall\, y \neq x.$$

Let $\tau_0 = 1$, and

$$Z_n = X(\tau_n),\ \ n = 0, 1, 2, \cdots$$

(Only count the jump each time, but ignore the length of waiting time).

Then,

$\{Z_n\}_{n=0}^{\infty}$ is a Markov chain with Q as transition matrix.

Note:

$$T_y \stackrel{\text{def}}{=} \inf\{t \geqslant \tau_1 : X(t) = y\} < \infty$$

iff

$$T'_y \stackrel{\text{def}}{=} \inf\{n \geqslant 1 : Z_n = y\} < \infty \text{ (as Markov chain).}$$

$\therefore \rho_{xy}$ for $\{Z_n\}_{n=0}^{\infty}$ is the same as ρ_{xy} for $\{X(t)\}_{t \geqslant 0}$.

$\therefore$To check recurrent/transience,

we need only consider Q!

Example: In the birth & death process

$$Q = \begin{bmatrix} 0 & 1 & & \\ \frac{\mu_1}{\lambda_1+\mu_1} & 0 & \frac{\lambda_1}{\lambda_1+\mu_1} & \\ & \frac{\mu_2}{\lambda_2+\mu_2} & 0 & \frac{\mu_2}{\lambda_2+\mu_2} \\ & \ddots & \ddots & \ddots \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 & & & \\ q_1 & 0 & p_1 & & \\ & q_2 & 0 & p_2 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

It follows from Chapter 1 (P33) that the chain is recurrent **iff**

$$\sum_{n=1}^{\infty} \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} = \sum_{n=1}^{\infty} \frac{q_1 \cdots q_n}{p_1 \cdots p_n} = \infty.$$

- **Long-run behavior.**

Let $m_x \overset{\text{def}}{=} E_x(T_x)$ (the mean return time).

– Null recurrent: $m_x = \infty$

– Positive recurrent: $m_x < \infty$. In this case

$$\boxed{\pi(x) = \frac{1}{q_x m_x}.} \qquad (*)$$

Intuitive Proof of $(*)$:

– In $[0, t]$ for large t, the process will visit x for $\frac{t}{m_x}$ times and the average time staying at x (waiting time to jump way) per visit is $1/q_x$.

– The total time spent in x during $[0, t]$ is $\frac{t}{m_x} \cdot \frac{1}{q_x}$.

– The proportion of time spent in x is $\frac{1}{q_x m_x}$.

Note: Any MJP is **aperiodic**.

For an irreducible, positive recurrent MJP,

$$\lim_{t\to\infty} P_{xy}(t) = \pi(y) = \frac{1}{q_y m_y}, \quad x, y \in S.$$

The end of lectures