Chapter 2: Stationary Distribution

§2.1 Stationary Distribution

Motivation: Recall the two state MC with

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}, \quad 0 < p, q < 1.$$

We have shown (Chapter 1):

$$\lim_{n\to\infty} P(X_n = 0) = q/(p+q) \stackrel{\text{def}}{=} a,$$
$$\lim_{n\to\infty} P(X_n = 1) = p/(p+q) = 1 - a.$$

Denote $\pi = [a, 1-a]$ (limit distribution), i.e.

$$\pi = \lim_{n\to\infty} \underbrace{[P(X_n = 0), P(X_n = 1)]}_{\text{pdf of } X_n} = \lim_{n\to\infty} \pi_0 P^n, \qquad (*)$$

where $\pi_0 = [P(X_0 = 0), P(X_0 = 1)]$ is the initial distribution.

Note: Here π is independent of π_0.

We discuss two issues related to π:

- It is direct to verify

$$\pi P = \pi,$$

i.e. $$[\frac{q}{p+q}, \frac{p}{p+q}] \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = [\frac{q}{p+q}, \frac{p}{p+q}].$$

Hence by induction,

$$\pi P^n = \pi, \quad n = 1, 2, \cdots$$

It means that if the chain starts with X_0 with pdf π, then at any time $n = 1, 2, \cdots$, X_n has the same distribution as π.

Note: $(*)$ also directly implies

$$\pi = \lim_{n\to\infty} (\pi_0 P^{n-1}) P = \pi P.$$

- One can also show:

$$\lim_{n\to\infty} P^n = \begin{bmatrix} a & 1-a \\ a & 1-a \end{bmatrix} = \begin{bmatrix} \pi \\ \pi \end{bmatrix}.$$

Two ways:

(i) Diasonalize P. See Tutorial or Exercise.

(ii) Find

$$\lim_{n\to\infty} P^n(x,y) = \lim_{n\to\infty} P(X_n = y | X_0 = x)$$
$$= \lim_{n\to\infty} P_x(X_n = y).$$

As proved before, for $x = 0$ or 1

$\lim_{n\to\infty} P_x(X_n = 0) = a$ i.e. the 1^{st} column is a,
$\lim_{n\to\infty} P_x(X_n = 1) = 1 - a$ i.e. the 2^{nd} column is $1 - a$.

Observe: The fact that

$$\pi P = \pi = 1 \cdot \pi$$

means that π is the **left 1-eigenvector** of P.

Thus, we may also find the limit distribution π directly by solving

$$[u, v] \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = [u, v],$$

i.e.

$$[u, v] \begin{bmatrix} -p & p \\ q & -q \end{bmatrix} = [0, 0], i.e., \begin{bmatrix} -p & q \\ p & -q \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$(\because\ u \geqslant 0, v \geqslant 0, u+v = 1\ \therefore\ u = \frac{q}{p+q}, v = \frac{p}{p+q})$$

General Situaion:

If

$$\pi = \lim_{n\to\infty} \pi_0 P^n \quad \text{exists}$$

for some initial distribution π_0,

then π satisfies

$$\pi = \left(\lim_{n\to\infty} \pi_0 P^{n-1}\right) P = \pi P,$$

i.e.,

$$\pi = \pi P,$$

or equivalently,

$$\pi(y) = \sum_{x\in S} \pi(x) P(x,y), \quad \forall\, y \in S.$$

Definition: We say that a probability row-vector π is a **stationary distribution** for P **if**

$$\pi = \pi P,$$

i.e. the pdf π is a left 1-eigenvector of P.

Two basic questions:

(i) **Existence ($\exists$):** Does every P have a SD?

(ii) **Uniqueness (!):** Is the SD unique?

Two notes:

(1) **If** $\pi = \pi P$ has a unique solut'n **then** the limit $\lim_{n\to\infty} \pi_0 P^n$ (if it exists) is independent of π_0.

(2) If $\lim_{n\to\infty} P^n = \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix}$ then for any initial distri π_0,

$$\lim_{n\to\infty} \pi_0 P^n = \pi,$$

i.e. the limit exists and is independent of π_0.

This also suggests a way of finding the SD of P.

Proposition: Let P be a Markov matrix with **finite** state space S. Assume:

(i) The left 1-eigenvector (which must exist) can be chosen to have all nonegative entries;

(ii) 1 is a simple eigenvalue;

(iii) other eigenvalues $|\lambda_i| < 1$.

Then P has a unique SD π, i.e. $\pi P = \pi$, and

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix}.$$

Pf.: (Sketch only) (See Lawler P11-15)

$$P = QDQ^{-1}, \quad D = \left[\begin{array}{c|c} 1 & O \\ \hline O & M \end{array}\right], \quad M^n \to 0$$

Q: columns are right eigenvectors; 1st row is $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

Q^{-1}: rows are left eigenvectors; 1st row is a prob vector, denoted by π

$$\therefore \lim_{n\to\infty} P^n = \lim_{n\to\infty} QD^nQ^{-1} = Q \left[\begin{array}{c|c} 1 & O \\ \hline O & O \end{array}\right] Q^{-1} = \begin{bmatrix} \pi \\ \vdots \\ \pi \end{bmatrix}$$

□

Remarks:

(1) **If** 1 is NOT a simple eigenvalue, **then** π may not be unique: e.g.

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

$$\left.\begin{array}{l} \pi_1 \text{ is the SD of } P_1 \\ \pi_2 \text{ is the SD of } P_2 \end{array}\right\} \Rightarrow [\lambda\pi_1, (1-\lambda)\pi_2] \text{ is the SD of } P$$

(2) **Without** (iii), the limit $\lim_{n\to\infty} P^n$ may not exist (but π still may exist): e.g.

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \pi = [\frac{1}{2}, \frac{1}{2}],$$

BUT eigenvalues: ± 1.

(3) Two further **Facts** for **finite** S (without proof):

Fact a. If for some $n \geq 1$, P^n has all entries strictly positive, then three conditions are satisfied, therefore, P has a unique SD π, and

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \pi \\ \vdots \\ \pi \end{bmatrix}.$$

Fact b. If P is irreducible, then P still has a unique SD.

(But $\lim_{n\to\infty} P^n$ may not exist, e.g., $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$)

Computation Technique for **finite** S:

Case 1: P is irreducible.

$$\pi P = \pi, \text{i.e., } P^T \pi^T = \pi^T, \text{i.e. } (P^T - I)\pi^T = 0.$$

$$P^T - I \xrightarrow{\text{row operation}} \begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & * \end{bmatrix},$$

Upper diagonal form.

Fact b above assures that the solution exists uniquely. (**Note:** Find π as a prob row-vector)

Case 2. P is reducible.

For instance, let $S = C_1 \cup C_2 \cup S_T$.
Reordering S accordingly, write

$$P = \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ S_1 & S_2 & Q \end{bmatrix}, \quad P^n = \begin{bmatrix} P_1^n & 0 & 0 \\ 0 & P_2^n & 0 \\ S_{1n} & S_{2n} & Q^n \end{bmatrix}$$

$$i = 1, 2 : \lim_{n \to \infty} P_i^n = \begin{bmatrix} \pi_i \\ \vdots \\ \pi_i \end{bmatrix}, \pi_i : \text{SD of } P_i,$$

$$\lim_{n \to \infty} Q^n = 0,$$

(Chap1: For $y \in S_T$, $\lim_{n \to \infty} P^n(x, y) = 0$, $\forall x \in S$).

In fact, all eigenvalues of Q have moduli < 1.

$$\therefore \lim_{n\to\infty} P^n = \begin{bmatrix} \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_1 \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} \pi_2 \\ \vdots \\ \pi_2 \end{bmatrix} & 0 \\ A_1 & A_2 & \mathbf{0} \end{bmatrix},$$

$A_1 = \lim_{n\to\infty} S_{1n}$, $A_2 = \lim_{n\to\infty} S_{2n}$:
$A_1(x, y) =$ prob from $x \in S_T$ to $y \in C_1$ in the long run,
$A_2(x, y) =$ prob from $x \in S_T$ to $y \in C_2$ in the long run.

Q.: How to find A_1, A_2?

Solution: Assume

$$S_T = \{x_1, x_2, \cdots, x_\ell\}.$$

First find

$$\rho_{C_i}(x), \quad x \in S_T, \ i = 1, 2,$$

(**absorption prob** of C_i, i.e. prob to enter C_i).

Then, distribute according to π_i, e.g.

$$A_1 = \begin{bmatrix} \rho_{C_1}(x_1)\pi_1 \\ \rho_{C_1}(x_2)\pi_1 \\ \vdots \\ \rho_{C_1}(x_\ell)\pi_1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \rho_{C_2}(x_1)\pi_2 \\ \rho_{C_2}(x_2)\pi_2 \\ \vdots \\ \rho_{C_2}(x_\ell)\pi_2 \end{bmatrix}. \quad \square$$

Example 1. (Gambler's ruin chain) Let

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Show that

$$\lim_{n\to\infty} P^n = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{3}{4} & 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Solution: From P, one can check that

$$C_1 = \{0\},\ C_2 = \{4\},\ S_T = \{1, 2, 3\},$$

and

$$S = C_1 \cup C_2 \cup S_T.$$

After reordering,

$$P = \begin{array}{c} \\ 0 \\ 4 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 4 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix} \end{array}.$$

Set $P = \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \mathbf{0} \\ \hline \multicolumn{2}{c|}{S} & Q \end{array}\right]$. Then, $P^n = \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \mathbf{0} \\ \hline \multicolumn{2}{c|}{S_n} & Q^n \end{array}\right]$, and

$$\lim_{n\to\infty} S_n = A, \qquad \lim_{n\to\infty} Q_n = 0,$$

$$\lim_{n\to\infty} P^n = \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \mathbf{0} \\ \hline \multicolumn{2}{c|}{A} & \mathbf{0} \end{array}\right].$$

Need to find $A = A_{3\times 2}$:

Note that for $i \in S_T = \{1, 2, 3\}$, $j \in C_1 \cup C_2 = \{0, 4\}$,

$A(i,j) =$ prob that the chain starting at i eventually visits j

$= P_i(T_j < \infty) = \rho_{ij},$

$\rho_{ij} = P(i,j) + \sum_{k\in S_T} P(i,k)\rho_{kj}.$

Put in matrix form

$$A = S + QA \qquad \therefore \boxed{A = (I - Q)^{-1}S.}$$

Here,

$$S_{3\times 2} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \; Q_{3\times 3} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \; \therefore (I-Q)^{-1} = \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ 1 & 2 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix}.$$

$$\therefore A = (I - Q)^{-1}S = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

	0	4	1	2	3
0	1	0	0	0	0
4	0	1	0	0	0
1	$\frac{3}{4}$	$\frac{1}{4}$	0	0	0
2	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
3	$\frac{1}{4}$	$\frac{3}{4}$	0	0	0

$\xrightarrow{\text{reorder}}$

	0	1	2	3	4
0	1	0	0	0	0
1	$\frac{3}{4}$	0	0	0	$\frac{1}{4}$
2	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$
3	$\frac{1}{4}$	0	0	0	$\frac{3}{4}$
4	0	0	0	0	1

$= \lim_{n\to\infty} P^n$.

□

Remark: Such computation also gives us a way to find

$$\rho_{10} = \frac{3}{4},\ \rho_{20} = \frac{1}{2},\ \rho_{30} = \frac{1}{4},$$
$$\rho_{14} = \frac{1}{4},\ \rho_{24} = \frac{1}{2},\ \rho_{34} = \frac{3}{4}.$$

Exercise: (Tutorial)

Modify the above to the MC with

$$P = \left[\begin{array}{cc|c|ccc} \frac{1}{3} & \frac{2}{3} & \mathbf{0} & & \mathbf{0} & \\ \frac{1}{2} & \frac{1}{2} & & & & \\ \hline & \mathbf{0} & 1 & & \mathbf{0} & \\ \hline \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{array}\right]$$

and find $\lim_{n\to\infty} P^n$.

Example 2. Consider the random walk on

$$S = \{0, 1, 2, \cdots\} \quad \text{(no longer finite!)}$$

with

$$P = \begin{bmatrix} q & p & & & \\ q & 0 & p & & \\ & q & 0 & p & \\ & & \ddots & \ddots & \ddots \end{bmatrix}, \quad p, q > 0, p + q = 1.$$

Q.: Find the SD.

Note:

- This is an irreducible BD chain.
- The chain is recurrent iff $\sum_{k=0}^{\infty} (\frac{q}{p})^k = \infty$, iff $q \geq p$.

Solution: Let π be the SD. Set

$$x_k = \pi(k), \quad k = 0, 1, \cdots$$

From $\pi = \pi P$, i.e.,

$$[x_0, x_1, \cdots] = [x_0, x_1, \cdots] \begin{bmatrix} q & p & & & \\ q & 0 & p & & \\ & q & 0 & p & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

we get

$$\begin{cases} x_0 = qx_0 + qx_1, \quad \text{i.e. } px_0 = qx_1, \\ k \geq 1 : x_k = qx_{k+1} + px_{k-1}. \end{cases}$$

$$\therefore qx_{k+1} - px_k = qx_k - px_{k-1} = \cdots = qx_1 - px_0 = 0$$

$$\therefore x_k = (\frac{p}{q})x_{k-1} = \cdots = (\frac{p}{q})^k x_0, \; k = 0, 1, 2, \cdots$$

(i) $p < q$ (recurrent):

$$1 = \sum_{k=0}^{\infty} x_k = \sum_{k=0}^{\infty} (\frac{p}{q})^k x_0 = \frac{x_0}{1 - \frac{p}{q}} \qquad (0 < \frac{p}{q} < 1)$$

$$\therefore x_0 = \frac{q - p}{q} > 0$$

SD: $\pi = \frac{q-p}{q}[1, \frac{p}{q}, (\frac{p}{q})^2, \cdots]$.

(ii) $p = q$ (recurrent): $\sum_{k=0}^{\infty} (\frac{p}{q})^k = \infty$. π does not exist.

(iii) $p > q$ (transient): π does NOT exist. □

Exercise: Modify it to the general irreducible birth & death chain on $S = \{0, 1, \cdots\}$ with

$$P = \begin{bmatrix} r_0 & p_0 & & & \\ q_1 & r_1 & p_1 & & \\ & q_2 & r_2 & p_2 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \quad \text{row sum} = 1,$$

$$\text{all } p_i > 0, \text{ all } q_i > 0.$$

Q.: Find the SD π.

Example 3. Queueing model:

- In a telephone exchange, ξ_n denotes no of new calls coming in starting at time $n \geqslant 1$. $\{\xi_n\}_{n=1}^{\infty}$ is i.i.d. and has a Poisson distribution with rate $\lambda > 0$:

$$p_k = e^{-\lambda}\frac{\lambda^k}{k!}, k = 0, 1, 2, \cdots$$

- Suppose that each call has prob $q \overset{\text{def}}{=} 1 - p$ to finish in one unit time.

$$X_n \overset{\text{def}}{=} \text{no of calls in progress at time } n.$$

Q.: Find the transition prob and the SD.

Solution: To find

$$P(x, y) = P(X_{n+1} = y | X_n = x),$$

we consider

$$X_{n+1} = \xi_{n+1} + Y_{n+1}$$

with $Y_{n+1} \stackrel{\text{def}}{=}$ no of calls at time n that remain at time $n+1$.

Fact:

$$P(Y_{n+1} = z | X_n = x) = \binom{x}{z} p^z (1-p)^{x-z},$$

$$0 \leqslant z \leqslant k.$$

Note: $p =$**non-finish** prob, $q = 1 - p =$**finish** prob.

$$\begin{aligned}
\therefore P(x,y) &= P(X_{n+1} = y | X_n = x) \\
&= \sum_{z=0}^{x\wedge y} P(X_{n+1} = y, Y_{n+1} = z | X_n = x) \\
&= \sum_{z=0}^{x\wedge y} P(\xi_{n+1} = y - z, Y_{n+1} = z | X_n = x) \\
&= \sum_{z=0}^{x\wedge y} P(\xi_{n+1}=y-z) P(Y_{n+1}=z|X_n=x) \\
&= \sum_{z=0}^{x\wedge y} e^{-\lambda} \frac{\lambda^{y-z}}{(y-z)!} \binom{x}{z} p^z (1-p)^{x-z}.
\end{aligned}$$

To find SD, we will verify that if X_0 is Poisson then X_n $(n \geqslant 1)$ satisfy the same Poisson distribution.

Lemma 1. If X_n is Poisson with rate t, then Y_{n+1} is Poisson with rate pt.

Pf.:

$$
\begin{aligned}
P(Y_{n+1} = y) &= \textstyle\sum_{x=y}^{\infty} P(Y_{n+1} = y, X_n = x) \\
&= \sum_{x=y}^{\infty} P(X_n = x) P(Y_{n+1} = y | X_n = x) \\
&= \sum_{x=y}^{\infty} e^{-t} \frac{t^x}{x!} \binom{x}{y} p^y (1-p)^{x-y} \\
&= \frac{(pt)^y e^{-t}}{y!} \sum_{x=y}^{\infty} \frac{[t(1-p)]^{x-y}}{(x-y)!} \\
&= \frac{(pt)^y e^{-t}}{y!} e^{t(1-p)} \\
&= e^{-pt} \frac{(pt)^y}{y!}, \quad y = 0, 1, 2, \cdots \qquad \square
\end{aligned}
$$

Lemma 2. If X, Y are independent Poisson with rates t_1 and t_2 resp, then $Z = X + Y$ is Poisson with rate $t_1 + t_2$.

Pf.:

$$\begin{aligned}
P(Z = z) &= P(X + Y = z) \\
&= \textstyle\sum_{x=0}^{z} P(X + Y = z, X = x) \\
&= \textstyle\sum_{x=0}^{z} P(X = x, Y = z - x) \\
&= \textstyle\sum_{x=0}^{z} P(X = x)P(Y = z - x) \\
&= \sum_{x=0}^{z} e^{-t_1} \frac{t_1^x}{x!} e^{-t_2} \frac{t_2^{z-x}}{(z-x)!} \\
&= \frac{e^{-(t_1+t_2)}}{z!} \sum_{x=0}^{z} \binom{z}{x} t_1^x t_2^{z-x} \\
&= \frac{e^{-(t_1+t_2)}}{z!} (t_1 + t_2)^z, \quad z = 0, 1, \cdots \quad \square
\end{aligned}$$

Two lemmas above give:

- Assume X_0 is Poisson with rate t (**TBD**).
- $X_1 = \xi_1 + Y_1$ is Poisson with rate
$$\lambda + pt = t. \quad (\therefore t \stackrel{\text{def}}{=} \frac{\lambda}{1-p} = \frac{\lambda}{q})$$
- $X_2 = \xi_2 + Y_2$ is Poisson with rate $\lambda + pt = t$.
- $\cdots$
- $X_n = \xi_n + Y_n$ is Poisson with rate $\lambda + pt = t$.

$\therefore$ **The chain has a SD** (Poisson, rate$= \lambda/q$):

$$\pi(x) = e^{-\lambda/q}\frac{(\lambda/q)^x}{x!}, \quad x = 0, 1, \cdots . \quad \square$$

Exercise: Check the textbook (Page 55-56) to

(i) Derive an explicit formular $P^n(x,y)$.

(ii) Show directly that

$$\lim_{n\to\infty} P^n(x,y) = \pi(y), \quad \forall\, x,y \geqslant 0.$$

(Hence, π that we have found is the **unique** SD)

Sketch: • The key is to find P^n:

$$
\begin{aligned}
X_0 &: \quad t \\
X_1 &: \quad \lambda + tp \\
X_2 &: \quad \lambda + (\lambda + pt)p = tp^2 + \lambda(1 + p^2) \\
X_3 &: \quad \lambda + [tp^2 + \lambda(1 + p^2)]p = tp^3 + \lambda(1 + p^2 + p^3) \\
\dots &\quad \dots \\
X_n &: \quad tp^n + \lambda(1 + p + \cdots + p^n) \\
&\quad = tp^n + \lambda \frac{1 - p^n}{1 - p} := t_n
\end{aligned}
$$

then

$$\sum_{x=0}^{\infty} e^{-t} \frac{t^x}{x!} P^n(x, y) = P_x(X_n = y) = e^{-t_n} \frac{t_n^y}{y!}.$$

Rewrite it as

$$\sum_{x=0}^{\infty} \frac{P^n(x,y)}{x!} t^x = e^{-\lambda \frac{1-p^n}{1-p}} e^{t(1-p^n)} \frac{\left[tp^n + \lambda \frac{1-p^n}{1-p}\right]^y}{y!},$$

Apply Taylor expansion and binormial expansion on the right, do the product, and compare coefficient of t^x for each x, then

$$P^n(x,y) = e^{-\lambda \frac{1-p^n}{1-p}} \sum_{z=0}^{\min(x,y)} \binom{x}{z} p^{nz} (1-p^n)^{x-z} \frac{\left[\lambda \frac{1-p^n}{1-p}\right]^{y-z}}{(y-z)!}.$$

Let $n \to \infty$, note $p^n \to 0$ as $0 \le p < 1$, in $\sum$, except for the term of $z = 0$, all other terms tend to zero, then

$$\lim_{n\to\infty} P^n(x,y) = e^{-\frac{\lambda}{1-p}} \frac{(\frac{\lambda}{1-p})^y}{y!} = \pi(y). \quad \square$$

§2.2 Average number of visits

Given $\{X_n\}_{n=0}^{\infty}$, S (finite or infinite),

$$N_n(y) \overset{\text{def}}{=} \text{no of visits to } y \ \underline{\text{in } n\text{-steps}},$$

$$\text{i.e. during times } m = 1, 2, \cdots, n.$$

We are interested in the limits of

$$\frac{N_n(y)}{n}, \quad \frac{E_x(N_n(y))}{n} \quad \text{as} \quad n \to \infty.$$

Note:

- $\frac{N_n(y)}{n}$: **proportion** of the first n units of time that the chain visits y, or **average no** of visits to y per unit time.
- $\frac{E_x(N_n(y))}{n}$: **expected proportion** for a chain starting at x, or **frequency** that the chain visits y from x.

It is direct to see

$$N_n(y) = \sum_{m=1}^{n} 1_y(X_m),$$

$$E_x(N_n(y)) = \sum_{m=1}^{n} P^m(x, y).$$

Case: y is transient.

$$\text{Recall: } P_x(N(y) < \infty) = 1.$$

$$\lim_{n\to\infty} N_n(y) = N(y) < \infty \text{ with prob 1},$$

$$\lim_{n\to\infty} E_x(N_n(y)) = E_x(N(y)) = \frac{\rho_{xy}}{1-\rho_{yy}} < \infty.$$

So,

$$\lim_{n\to\infty} \frac{N_n(y)}{n} = 0 \quad \text{with prob 1},$$

$$\lim_{n\to\infty} \frac{E_x(N_n(y))}{n} = 0.$$

Hence, we only consider y **as a recurrent state**.

Let y be **recurrent**. Denote

$$m_y \stackrel{\text{def}}{=} E_y(T_y): \quad \text{the mean return time to } y \text{ for a chain starting at } y.$$

Recall

$$T_y \stackrel{\text{def}}{=} \min\{n \geqslant 1 : X_n = y\}.$$

Theorem: Suppose

$$\{X_n\}_{n=0}^{\infty} \text{ is } \textbf{irreducible} \text{ and } \textbf{recurrent}.$$

Then

$$\lim_{n\to\infty} \frac{N_n(y)}{n} = \frac{1}{m_y} \quad \text{with prob 1,}$$

$$\lim_{n\to\infty} \frac{E_x(N_n(y))}{n} = \frac{1}{m_y}, \quad \forall\, x \in S.$$

Remarks:

(1) Heuristically, the limit is the **frequency** and m_y is the **waiting time**. They are reciprocal to each other.

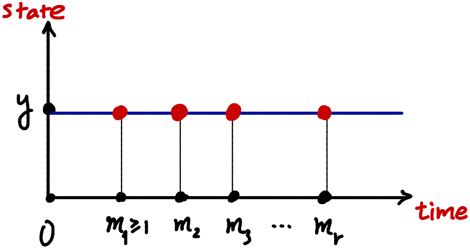
(2) If the chain is NOT irreducible, the statement of Theorem can be modified slightly; see the textbook Pages 58-59.

Pf.: Let the chain start from y. Introduce new r.v.:

$$T_y^r = \min\{n \geqslant 1 : N_n(y) = r\}, \quad r = 1, 2, \cdots$$

i.e. the min ptv-time of the r^{th} visit to y. **Note:**

- $N_n(y) = r$: By time n, the chain visits y for r times. (Warning: time 0 not counted).
- T_y^r: the min positive time up to which the chain visits y for exactly r times.



$$\mathbb{1}_y(X_i) = \begin{cases} 1 & i = m_1, m_2, \cdots, m_r \\ 0 & \text{otherwise} \end{cases}$$

Set

$$W^1 \stackrel{\text{def}}{=} T_y^1 = T_y \text{ (i.e. hitting time of } y\text{)}$$

$$W^r \stackrel{\text{def}}{=} T_y^r - T_y^{r-1}, \quad r = 2, 3, \cdots$$

(i.e., waiting time between the $(r-1)^{th}$ visit to y

and the r^{th} visit to y)

Then

$$T_y^r = W_y^1 + \cdots + W_y^r, \quad r = 1, 2, \cdots$$

Note: $\{W_y^r\}_{r=1}^{\infty}$ is i.i.d.

(it is intuitively obvious due to the Markov property; see the textbook (page 59) for the rigorous proof)

Apply the **SLLN**, we have

$$\lim_{r\to\infty} \frac{T^r}{r} = \lim_{r\to\infty} \frac{W_y^1+\cdots+W_y^r}{r} = E_y(T_y) \text{ with prob } 1$$
$$= m_y.$$

Next, let $r = N_n(y)$, i.e. by time n, the chain visits y for r-times, and the $(r+1)^{th}$ visit to y will be after n, hence

$$T_y^r \leqslant n < T_y^{r+1},$$

so that

$$\frac{T_y^r}{r} \leqslant \frac{n}{N_n(y)} = \frac{n}{r} < \frac{T_y^{r+1}}{r} \to m_y \quad \text{as } r \to \infty.$$

This implies that $\lim_{n\to\infty} \frac{n}{N_n(y)} = m_y$ with prob 1. □

Moreover, we observe

$$\lim_{n\to\infty} E_x\left(\frac{N_n(y)}{n}\right) = E_x\left(\lim_{n\to\infty}\frac{N_n(y)}{n}\right) \quad \text{(why? DCT)}$$
$$= E_x\left(\frac{1}{m_y}\right)$$
$$= \frac{1}{m_y}. \quad \square$$

Added: **Theorem** (Dominated convergence theorem). Let (ξ_n) be a sequence of rv's and ξ be a rv s.t. for each $\omega \in \Omega$, $\xi_n(\omega) \to \xi(\omega)$ as $n \to \infty$, and there is a rv η such that $|\xi_n| \le \eta$ and $E(\eta) < \infty$. Then

$$E|\xi_n - \xi| \to 0 \text{ as } n \to \infty.$$

Particularly,

$$E(\xi_n) \to E(\xi) \text{ as } n \to \infty. \quad \square$$

Remark: The statement of the theorem can be slightly modified in case when the chain is not irreducible. Indeed, for a general MC, as long as y is recurrent,

$$\frac{N_n(y)}{n} = \frac{\sum_{m=1}^{n} 1_y(X_m)}{n} \to \frac{1_{\{T_y<\infty\}}}{m_y} \text{ as } n \to \infty \text{ with prob } 1,$$

$$E_x(\frac{N_n(y)}{n}) = \frac{\sum_{m=1}^{n} P^m(x,y)}{n} \to \frac{\rho_{xy}}{m_y} \text{ as } n \to \infty,$$

where $1_{\{T_y<\infty\}}$ is a rv meaning that $1_{\{T_y<\infty\}} = 1$ if $T_y < \infty$, and $1_{\{T_y<\infty\}} = 0$ if $T_y = \infty$.

§2.3 Waiting time & stationary distribution

Def.:

- A state x is called **positive recurrent if** it is recurrent and $m_x = E_x(T_x) < \infty$.
- x is called **null recurrent if** it is recurrent and $m_x = E_x(T_x) = \infty$.

Note:

- For a null recurrent sate x,

$$\lim_{n\to\infty} \frac{N_n(x)}{n} = 0 \text{ with prob } 1, \quad \lim_{n\to\infty} \frac{E_x(N_n(x))}{n} = 0.$$

- A positive recurrent state means it comes back in finite waiting time; a null recurrent means it comes back very rarely.

THREE Theorems and THREE Corollaries
are COMING soon.....

no worry

Theorem 1. If x is positive recurrent and $x \to y$, then y is also positive recurrent.

Pf.: $\because x \to y$
$\therefore P^{n_1}(x,y) > 0$ for some $n_1 \geqslant 1$
$\because x$ recurrent, $x \to y$
$\therefore y \to x$, then $P^{n_2}(y,x) > 0$ for some $n_2 \geqslant 1$.
Hence

$$P^{n_2+m+n_1}(y,y) \geqslant P^{n_2}(y,x)P^m(x,x)P^{n_1}(x,y).$$

Sum over $m = 1, 2, \cdots, n$, and divide by n:

$$\frac{E_y(N_{n_2+n+n_1}(y)) - E_y(N_{n_2+n_1}(y))}{n} \geqslant P^{n_2}(y,x)\frac{E_x(N_n(x))}{n}P^{n_1}(x,y).$$

Take limit $n \to \infty$:

$$\frac{1}{m_y} \geqslant P^{n_2}(y,x)\frac{1}{m_x}P^{n_1}(x,y) > 0.$$

$\therefore m_y < \infty$, i.e. y is positive recurrent. □

Theorem 2. An irreducible MC having a finite number of states must be positive recurrent.

Pf.: We know that all states are recurrent ($\because$ finite state + irreducible).

Assuming that the theorem is false, all states are null recurrent. Note

$$1 = \sum_{y \in S} P^m(x, y) \qquad \text{(row sum is 1).}$$

Sum over $m = 1, \cdots, n$ and divide by n:

$$1 = \sum_{y \in S} \frac{E_x(N_n(y))}{n}.$$

Take limit:

$$
\begin{aligned}
1 &= \lim_{n\to\infty} \sum_{y\in S} \frac{E_x(N_n(y))}{n} \\
&= \sum_{y\in S} \lim_{n\to\infty} \frac{E_x(N_n(y))}{n} \quad \textcolor{red}{(S \text{ is finite})} \\
&= \sum_{y\in S} 0 \\
&= 0, \ \textcolor{red}{\text{contradiction!}} \quad \square
\end{aligned}
$$

Theorem 3. An irreducible positive recurrent MC has a unique SD π given by

$$\pi(x) = \frac{1}{m_x}, \quad x \in S.$$

Pf.: Step 1. Uniqueness.

We first assume the SD exists, denoted by π, to show $\pi(x) = \frac{1}{m_x}$, $x \in S$. In fact,

$$\pi(x) = \sum_z \pi(z) P^m(z,x) \quad (i.e., \pi = \pi P^m, \forall\, m \geqslant 1)$$

Sum over $m = 1, \cdots, n$ and divide by n:

$$\pi(x) = \sum_z \pi(z) \frac{E_z(N_n(x))}{n}.$$

Take limit:

$$\begin{aligned}\pi(x) &= \lim_{n\to\infty} \sum_z \pi(z) \frac{E_z(N_n(x))}{n} \\ &= \sum_z \pi(z) \lim_{n\to\infty} \frac{E_z(N_n(x))}{n} \ \big(\text{infinite sum need DCT}\big) \\ &= \sum_z \pi(z) \frac{1}{m_x} \\ &= \frac{1}{m_x}.\end{aligned}$$

Therefore, the uniqueness follows.

Added: **Dominated Convergence Theorem:** Suppose

(i) $|a_n(k)| \leqslant M < \infty$, $\lim_{n\to\infty} a_n(k) = a(k)$.

(ii) $\sum_{k=1}^{\infty} p_k = 1$ (or just $< \infty$)

Then

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_n(k)p(k) = \sum_{k=1}^{\infty} a(k)p(k).$$

(e.g. $\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{n+k}$ exists or not?)

Pf.: Apply ϵ-N argument to

$$\underbrace{\sum_{k=1}^{N} a_n(k)p_k}_{\text{(I)}} + \underbrace{\sum_{k=N+1}^{\infty} a_n(k)p_k}_{\text{(II)}}$$

(II)$\leq \epsilon/2$ for a large N.

(I): can be close to $\sum_{k=1}^{N} a(k)p(k)$ as long as n is large! □

Step 2. Existence.

To show existence, it suffices to show

(i) $\sum_{x \in S} \frac{1}{m_x} = 1$. (distribution)

(ii) $\sum_{x \in S} \frac{1}{m_x} P(x, y) = \frac{1}{m_y}, \forall\, y$. (stationary)

Step 2.1 To show: (i) (ii) are two <u>inequalities</u> "$\leqslant$".

- Note: $\sum_{x} P^m(z,x) = 1, \ \forall z$. Then

$$\frac{\sum_{m=1}^{n}(\cdots)}{n} \Rightarrow \sum_{x\in S} \frac{E_z(N_n(x))}{n} = 1 \text{ (if } S \text{ is infinite, why? Fubini!)}$$

(It will be direct if we take limit on n then "$=$" will follow, however, we cannot apply DCT here (why?) we need a slight modification)

$$\Rightarrow \sum_{x\in S_1} \frac{E_z(N_n(x))}{n} \leqslant 1, \forall S_1 \text{ finite}$$

$$\Rightarrow \sum_{x\in S_1} \frac{1}{m_x} \leqslant 1, \forall S_1 \text{ finite} \Rightarrow \sum_{x\in S} \frac{1}{m_x} \leqslant 1.$$

- Note

$$\sum_{x\in S} P^m(z,x)P(x,y) = P^{m+1}(z,y).$$

$\sum_{m=1}^{n}(\cdots)/n \Rightarrow$

$$\sum_{x\in S} \frac{E_z(N_n(x))}{n} P(x,y) = \frac{E_z(N_{n+1}(y))}{n} - \frac{P(z,y)}{n}.$$

$$\Rightarrow \sum_{x\in S_1} \frac{E_z(N_n(x))}{n} P(x,y) \leqslant \frac{E_z(N_{n+1}(y))}{n} - \frac{P(z,y)}{n}, \forall\, S_1 \text{ finite}$$

$$\Rightarrow \sum_{x\in S_1} \frac{1}{m_x} P(x,y) \leqslant \frac{1}{m_y}, \forall\, S_1 \text{ finite}$$

$$\Rightarrow \sum_{x\in S} \frac{1}{m_x} P(x,y) \leqslant \frac{1}{m_y}.$$

Step 2.2 To show: (i) (ii) are two equalities "$=$".

To show: (ii) $\sum_{x\in S} \frac{1}{m_x} P(x,y) = \frac{1}{m_y}$, $\forall\, y$.

Otherwise, $\exists\, y_0$ s.t.

$$\sum_{x\in S} \frac{1}{m_x} P(x, y_0) < \frac{1}{m_{y_0}}.$$

Then

$$\begin{aligned} 1 \geq \sum_{y\in S} \frac{1}{m_y} &> \sum_{y\in S} \left[\sum_{x\in S} \frac{1}{m_x} P(x,y) \right] \\ &= \sum_{x\in S} \frac{1}{m_x} \left[\sum_{y\in S} P(x,y) \right] \qquad \text{(Use Fubini)} \\ &= \sum_{x\in S} \frac{1}{m_x} \qquad \text{a contradiction!} \end{aligned}$$

To show (i): $\sum_{x \in S} \frac{1}{m_x} = 1$.

Note $\sum_{x \in S} \frac{1}{m_x} \leq 1$. Let c be such that $\sum_{x \in S} \frac{c}{m_x} = 1$. Then

$$\pi(x) = \frac{c}{m_x}, \quad x \in S$$

is a SD. Now, by uniqueness

$$\frac{c}{m_x} = \frac{1}{m_x}, \qquad \forall x \in S.$$

$\therefore c = 1$. So $\sum_{x \in S} \frac{1}{m_x} = 1$, i.e. (i) follows. $\square$

Corollary 1. An irreducible MC with finite state space has a unique SD:

$$\pi(x) = \frac{1}{m_x}, \quad x \in S.$$

e.g.: P (finite Matrix). $\pi P = \pi$. We then solve

$$(P^T - I)\pi^T = 0$$

though the row operation. **Cor 1** says that

- the solution exists and is unique.
- it gives us a way to find $m_x = E_x(T_x)$:

$$m_x = \frac{1}{\pi(x)}, \quad x \in S.$$

$$(\because\ m_x < \infty\ \ \therefore\ \pi(x) > 0)$$

Corollary 2. Let the chain be irreducible, then the chain has a SD **iff** it is positive recurrent!

Pf.: "$\Leftarrow$": It's just the theorem.

"$\Rightarrow$": Otherwise, all states are either null recurrent or transient (why?), then in both cases,

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} P^m(z,x)}{n} = 0, \quad \forall\, z, x \in S.$$

Let π be the SD. Take $x \in S$, then

$$\pi(x) = \sum_{z} \pi(z) P^m(z,x).$$

$$\sum_{m=1}^{n} (\cdots)/n \Rightarrow \pi(x) = \sum_{z} \pi(z) \frac{E_z(N_n(x))}{n}.$$

$$n \to \infty + \text{DCT} \Rightarrow \pi(x) = \sum_{z} \pi(z) \cdot 0 = 0. \quad \text{Contradiction!} \quad \square$$

e.g.: $P = \begin{bmatrix} q & p & & & \\ q & 0 & p & & \\ & q & 0 & p & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$, $S = \{0, 1, 2, \cdots\}$ is infinite.

Assume: $p > 0$, $q > 0$, $p + q = 1$ (irreducible).

Recall:

- This chain is recurrent **iff** $q \geqslant p$.
- The chain has a SD **iff** $q > p$.

Then, the chain is positive recurrent **iff**

$$q > p.$$

(Once again, in this case, $E_x(T_x) = m_x = \frac{1}{\pi(x)}$.)

Exercise: Consider a general birth & death chain

$$P = \begin{bmatrix} r_0 & p_0 & & & \\ q_1 & r_1 & p_1 & & \\ & q_2 & r_2 & p_2 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \qquad \text{Row sum} = 1$$

Assume it is irreducible.

Q.: Determine if it is either positive recurrent, null recurrent, or transient.

Corollary 3. Let C be an irreducible closed set of positive recurrent states. Then the MC has a unique SD π concentrated on C:

$$\pi(x) = \begin{cases} \frac{1}{m_x} & x \in C, \\ 0 & \text{Otherwise.} \end{cases}$$

Indeed, we can regard $\{X_n\}$ as a MC on C and obtain $\pi_C(x) = \frac{1}{m_x}$, $x \in C$. Define

$$\pi(x) = \begin{cases} \pi_C(x) & x \in C, \\ 0 & \text{Otherwise.} \end{cases}$$

Then it is direct to check that π is a SD on S.

e.g.: Let $S = (C_1 \cup \cdots) \cup S_T$ (finite or ∞), and

$$P = \begin{array}{c} \\ C_1 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cc} C_1 & \cdots \end{array} \\ \begin{bmatrix} P_1 & 0 \\ * & * \end{bmatrix} \end{array}, \quad C_1 \text{ positive recurrent.}$$

Regard $\{X_n\}_{n=0}^{\infty}$ as a MC on C_1. Then, by Thm 3, $\pi_{C_1}(x) = \frac{1}{m_x}$ $(x \in C_1)$ is the SD. Define

$$\pi(x) = \begin{cases} \pi_{C_1}(x) & \text{if } x \in C_1, \\ 0 & \text{Otherwise.} \end{cases}$$

We may write $\pi = [\pi_{C_1}, 0]$. Check:

$$\pi P = [\pi_{C_1}, 0] \begin{bmatrix} P_1 & 0 \\ * & * \end{bmatrix} = [\pi_{C_1} P_1, 0] = [\pi_{C_1}, 0] = \pi,$$

i.e. π is a SD of P.

Two further notes:

- If no C_i is positive recurrent (i.e. all states in S are either transient or null recurrent), then the chain has no SD.
- Let $P = \begin{array}{c} \\ C_1 \\ C_2 \\ S_T \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & S_T \end{array} \\ \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ * & * & * \end{bmatrix} \end{array}$,

  C_i $(i = 1, 2)$: positive recurrent,
  π_i $(i = 1, 2)$: SD of P_i concentrated on C_i.
  Then

$$\pi \overset{\text{def}}{=} \lambda\pi_1 + (1-\lambda)\pi_2,\ 0 \leqslant \lambda \leqslant 1$$

  is also the SD of P.

§2.4 Periodicity

Recall: For an irreducible & positive recurrent MC,

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} p^m(x,y)}{n} = \lim_{n\to\infty} \frac{E_x(N_n(y))}{n} = \frac{1}{m_y} = \pi(y), \ \forall\, y \in S,$$

$$\text{i.e.} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} P^m = \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix} \ \text{exist}$$

(S: finite or infinite)

Q.: How about $\lim_{n\to\infty} P^n$?

Example: $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, SD: $\pi = [\frac{1}{2}, \frac{1}{2}]$. Note:

$$P^{2n} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P^{2n+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\therefore \lim_{n \to \infty} P^n$ does NOT exist.

BUT, both $\lim_{n \to \infty} P^{2n}$ and $\lim_{n \to \infty} P^{2n+1}$ exist!

The problem is on the "**periodicity**" of the chain.

Definition. The period d_x of a state x is the **greatest common divisor** (**g.c.d.**) of

$$\{n \geqslant 1 : P^n(x,x) > 0\}.$$

Remarks:

(i) $1 \leqslant d_x \leqslant \min\{n \geqslant 1, P^n(x,x) > 0\}$.

(ii) If $P(x,x) > 0$ then $d_x = 1$.

(iii) For Example above, $d_0 = 2 = d_1$. Indeed, note:

$$1 = P^2(0,0) = P^4(0,0) = \cdots = P^{2n}(0,0) = \cdots ,$$
$$0 = P^1(0,0) = P^3(0,0) = \cdots = P^{2n+1}(0,0) = \cdots ,$$

$$\therefore \text{ g.c.d.}\{n \geq 1 : P^n(0,0) > 0\} = \text{g.c.d.}\{2, 4, \cdots\} = 2.$$

Prop. For an irreducible MC, all d_x are equal.

Pf.: T ake $x, y \in S$.
$\because$ The chain is irreducible
$\therefore x \to y$ & $y \to x$,
i.e. $\exists n_1 \geq 1, n_2 \geq 1$ s.t. $P^{n_1}(x,y) > 0$, $P^{n_2}(y,x) > 0$
So

$$P^{n_1+n_2}(x,x) \geqslant P^{n_1}(x,y)P^{n_2}(y,x) > 0$$

$\therefore d_x | n_1 + n_2$ $(*)$ (i.e., d_x is a divisor of $n_1 + n_2$)
Let $A_y \stackrel{\text{def}}{=} \{n \geqslant 1 : P^n(y,y) > 0\}$. Then, for $n \in A_y$,

$$P^{n_1+n+n_2}(x,x) \geqslant P^{n_1}(x,y)P^n(y,y)P^{n_2}(y,x) > 0$$

$\therefore d_x | n_1 + n + n_2$ Note: $n = (n_1 + n + n_2) - (n_1 + n_2)$
Together with $(*) \Rightarrow d_x | n$, $\forall\, n \in A_y$.

$$\therefore d_x | d_y$$

The same argument gives $d_y | d_x$. $\therefore d_x = d_y$. □

Definition: Consider an **irreducible** MC.

- Note that all states have

  the same period $d \geqslant 1$.

  The chain is called **periodic** with period $d \geqslant 1$.

- If $d = 1$, we say the chain is **aperiodic**.

Remark: Consider an irreducible MC. If

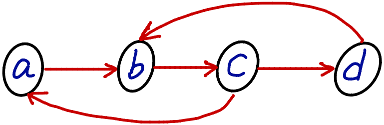
$$P(x, x) > 0 \text{ for some } x \in S,$$

then the chain must be aperiodic. ($\because$ $d_x = 1 = d$)

Example 1.

$$P = \begin{bmatrix} 0 & \times & 0 & 0 \\ 0 & 0 & \times & 0 \\ \times & 0 & 0 & \times \\ 0 & \times & 0 & 0 \end{bmatrix}, \quad \times\text{: nonzero entries.}$$

$p, q > 0;\ p + q = 1$



It is obvious to see that the chain is irreducible, and

$$d_a = 3,$$

(**Note:** $d_a = 3$ means that the chain from a returns to a in $3m$ steps, i.e. $P^{3m}(a,a) > 0,\ \forall\, m \geqslant 1$.)

$\therefore$ **Period** $= 3$.

We may directly compute: P^n, $(n = 2, 3, 4, \cdots)$.
For $m = 1, 2, \cdots,$

$$P^{3m} = \begin{bmatrix} \times & 0 & 0 & \times \\ 0 & \times & 0 & 0 \\ 0 & 0 & \times & 0 \\ \times & 0 & 0 & \times \end{bmatrix}, \quad P^{3m+1} = \begin{bmatrix} 0 & \times & 0 & 0 \\ 0 & 0 & \times & 0 \\ \times & 0 & 0 & \times \\ 0 & \times & 0 & 0 \end{bmatrix}$$

$$P^{3m+2} = \begin{bmatrix} 0 & 0 & \times & 0 \\ \times & 0 & 0 & \times \\ 0 & \times & 0 & 0 \\ 0 & 0 & \times & 0 \end{bmatrix}.$$

Recall: $d_x = \text{g.c.d. } \{n \geqslant 1 : P^n(x, x) > 0\}$.

$\therefore$ **Period** $= 3$.

Example 2. Determine the **period** of an irreducible birth and death chain:

$$P = \begin{bmatrix} r_0 & p_0 & & & \\ q_1 & r_1 & p_1 & & \\ & q_2 & r_2 & p_2 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}, \text{ all } p_x > 0, q_x > 0.$$

- If **some** $r_x > 0$, then $P(x,x) = r_x > 0$, hence the chain is aperiodic.
- If **all** $r_x = 0$, then the chain can return to its initial state ONLY after an even number of steps.
  Then, for a given state $x \in S$, any integer $n \geq 1$ such that $P^n(x,x) > 0$ must be even.
  Then $d \geqslant 2$ must be even.
  Note $P^2(0,0) = P(0,1)P(1,0) = p_0 q_1 > 0$.
  $\therefore$ **Period** $= 2$.

Theorem. Let $\{X_n\}_{n=0}^{\infty}$ be irreducible and positive recurrent with SD π.

(i) **If** the chain is **aperiodic**, **then**

$$\lim_{n\to\infty} P^n(x,y) = \pi(y), \quad \forall\, x, y \in S.$$

(ii) **If** the chain is **periodic** with period $d \geqslant 2$, **then** for any $x, y \in S$, there exists

$$r \in \{0, 1, 2, \cdots, d-1\}$$

which may depend on x and y, s.t.

$$P^n(x,y) = \begin{cases} \xrightarrow[m\to\infty]{} d\pi(y) & \text{if } n = md + r, \\ = 0 & \text{if } n \neq md + r, \end{cases}$$

where $m \geqslant 0$ is an integer.

Pf.: Pages 75-80 in the textbook. □

Remark: Theorem tells that in case

$$\textbf{Period} = d \geq 2,$$

we are able to determine the limits of

$$P^{md}, \quad P^{md+1}, \cdots, P^{md+(d-1)} \quad (m \to \infty).$$

Precisely, for any given x, y,

$$P^{md}(x,y), \quad P^{md+1}(x,y), \cdots, P^{md+(d-1)}(x,y)$$

are zeros, **except that**

exactly one of them tends to $d\pi(y)$ as $m \to \infty$.

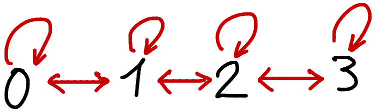
You have to figure out which one!

Example 3. Determine the long term behavior of P^n for given P.

(a) $$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}.$$

Solution:

- Note:



$\therefore$ irreducible.

- $\exists\, x$ s.t. $P(x,x) > 0$. $\therefore$ Period $= 1$.
- Solving $\pi = \pi P$, we get the ! SD

$$\pi = [\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}].$$

- Hence, by the theorem,

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \pi \\ \pi \\ \pi \\ \pi \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}. \quad \square$$

(b) $P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$

Solution:

- Note:

$$0 \leftrightarrow 1 \leftrightarrow 2 \leftrightarrow 3$$

$\therefore$ irreducible.

- Period $= 2$. (By the previous example)
- Solving $\pi = \pi P$, we get the ! SD: $\pi = [\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}]$.

$$0 \leftrightarrow 1 \leftrightarrow 2 \leftrightarrow 3$$

By the theorem,

if $x - y$ is even, $\begin{cases} P^{2m+1}(x,y) = 0, & \forall\, m, \\ P^{2m}(x,y) \xrightarrow[m\to\infty]{} 2\pi(y). \end{cases}$

If $x - y$ is odd, $\begin{cases} P^{2m}(x,y) = 0, & \forall\, m, \\ P^{2m+1}(x,y) \xrightarrow[m\to\infty]{} 2\pi(y). \end{cases}$

Recall: $\pi = [\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}]$.

$$\therefore \lim_{m\to\infty} P^{2m} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \hline \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \end{array}, \quad \lim_{m\to\infty} P^{2m+1} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \hline 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \end{array}.$$