Homework 1

1. Decide whether the following sets are convex and affine. Give the convex and affine hull if it is not convex or affine.

(a) A ray, which has the form $\{x \in \mathbb{R}^N : x = x_0 + \theta v, \theta \geq 0\}$, where $x_0 \in \mathbb{R}^N$ and $0 \neq v \in \mathbb{R}^N$ are fixed vectors.

(b) The $N-1$-dimension Euclidean unit disk in $\mathbb{R}^N$, which has the form

$$\left\{(x_1, x_2, \ldots, x_{N-1}, 0) : \sqrt{x_1^2 + x_2^2 + \ldots x_{N-1}^2} \leq 1\right\}$$

(c) The set of all solutions of the linear equation $Ax = b$ for some matrix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, *i.e.*,

$$\left\{x \in \mathbb{R}^N : Ax = b\right\},$$

where we assume the solution set is non-empty. (Hint: recall how to solve the non-homogeneous linear equations in linear algebra.)

2. Prove that the intersection of any number (perhaps infinite and uncountable) of convex sets is convex.

3. Prove that a cone C is convex if and only if $C + C \subset C$. (Hint: $\Leftarrow$ take $x, y \in C$ and prove that $\theta x + (1-\theta) y \in C$ for $\theta \in [0,1]$; $\Rightarrow$ take $x, y \in C$ and consider $\frac{x+y}{2}$).