Mid-term Examination

Partial Differential Equations (MATH4220)

(Academic Year 2023/2024, Second Term)

Date: March 14th, 2024. **Time allowed:** 8:30-10:15

Recall that, the solution u for 1D heat equation

$$\begin{cases} \partial_t u = \partial_x^2 u, & \text{in } (t,x) \in [0,\infty) \times \mathbb{R}_x, \\ u_{|t=0} = \phi(x), & \text{for } x \in \mathbb{R}_x, \end{cases}$$

is given by

$$u(t,x) = \int_{\mathbb{R}} S(t,x-y)\phi(y)\mathrm{d}y, \quad \text{where } S(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

For any $r > 0$, we denote

$$B_r = \left\{ x \in \mathbb{R}^3 : |x| \le r \right\} \quad \text{and} \quad \partial B_r = \left\{ x \in \mathbb{R}^3 : |x| = r \right\}.$$

1. Consider the following four questions.

   (a) (5 points) State the definition of a linear PDE problem.

   (b) (5 points) State the definition of a harmonic function.

   (c) (5 points) Solve the PDE $\partial_t u(t,x) - \partial_x u(t,x) = 0$ with $u(0,x) = x^4$.

   (d) (5 points) Solve the PDE $\partial_{xy} u(x,y) = 0$ where $u(x,y) : \mathbb{R}^2 \to \mathbb{R}$.

   (e) (5 points) Solve the PDE $\partial_t u(t,x) + x\partial_x u(t,x) = 0$ with $u(0,x) = x^4$.

2. (a) (5 points) Show that $\int_{\mathbb{R}} e^{-x^2} \mathrm{d}x = \sqrt{\pi}$.

   (b) (10 points) Show that the 1D heat equation $\partial_t u - \partial_x^2 u = 0$ is not well-posed for $t < 0$.

   Hint: Consider $u_n(t,x) = \frac{1}{n} e^{-n^2 t}(\sin nx)$.

   (c) (10 points) Consider the following initial-boundary value problem

$$\begin{cases} \partial_t u - \partial_x^2 u = f(t,x), & (t,x) \in [0,\infty) \times [0,L], \\ u(0,x) = \phi(x), \\ u(t,0) = g(t) \quad \text{and} \quad u(t,L) = h(t). \end{cases}$$

   Show that the uniqueness of the above problem by energy method.

   (d) (10 points) Solve the 1D heat equation with variable dissipation:

$$\partial_t u - \partial_x^2 u + 7t^6 u = 0, \quad \text{for } (t,x) \in [0,\infty) \times \mathbb{R}_x \quad \text{with } u(0,x) = \phi(x).$$

3. (5 points) Suppose that $v \in C^2(B_1) \cap C^1(\bar{B}_1)$ is a subharmonic function in B_1, that is, $\Delta v \geq 0$. Show that

$$\sup_{B_1} v \leq \sup_{\partial B_1} v.$$

Hint: Consider $v_\varepsilon(x) = v(x) + \varepsilon |x|^2$ for $\varepsilon > 0$.

4. Suppose that $u \in C^2(B_1) \cap C^1(\bar{B}_1)$ is a harmonic function in B_1.

(a) (5 points) Show that $\Delta(u^2) = 2|\nabla u|^2$.

(b) (10 points) Suppose that $\eta \in C_c^2(B_1)$ with $\eta \equiv 1$ in $B_{\frac{1}{2}}$. Show that

$$\begin{aligned} \Delta\left(\eta^2 |\nabla u|^2\right) &= 2\eta\Delta\eta|\nabla u|^2 + 2|\nabla\eta|^2|\nabla u|^2 \\ &\quad + 8\eta \sum_{i,j=1}^{3} \left(\partial_{x_i}\eta\right)\left(\partial_{x_j} u\right)\left(\partial_{x_i x_j} u\right) + 2\eta^2 \sum_{i,j=1}^{3} \left(\partial_{x_i x_j} u\right)^2. \end{aligned}$$

(c) (10 points) Show that

$$\Delta\left(\eta^2|\nabla u|^2\right) \geq \left(2\eta\Delta\eta - 6|\nabla\eta|^2\right)|\nabla u|^2.$$

(d) (5 points) Show that there exists $C > 0$ such that the function $(\eta^2|\nabla u|^2 + Cu^2)$ is a subharmonic function on B_1.

(e) (5 points) Conclude that

$$\sup_{B_{\frac{1}{2}}} |\nabla u| \leq \sqrt{C} \sup_{B_1} |u|.$$

*************** END OF THE QUESTIONS ***************