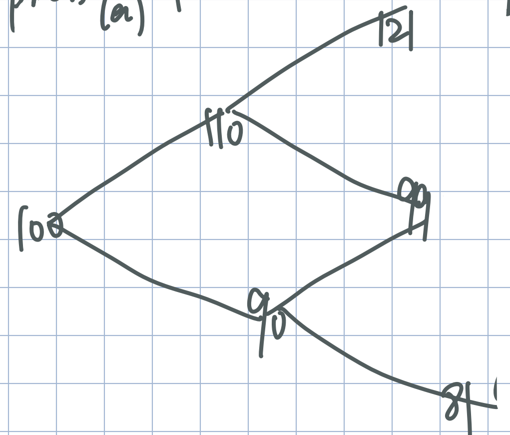
Problem 1.

(a)



expected payoff

0

0

0

9

$$q = \frac{1.03-0.9}{1.1-0.9} = 0.65$$

$$f_u = 1.03^{-1}(0.65\times 0 + 0.35\times 0) = 0$$

$$f_d = 1.03^{-1}(0.65\times 0 + 0.35\times 9) = 3.06$$

$$\phi_u = 0$$

$$\phi_d = \frac{0-9}{99-81} = -\frac{1}{2}$$

$$f = 1.03^{-1}(0.65\times 0 + 0.35\times 3.06) = 1.04$$

$$\phi = \frac{0-3.06}{110-90} = -0.15$$

strategy:

t_0: short 0.15 share of stock.

t_1: ↗ buy 0.15 shares to hold 0 share.

↘ short extra 0.35 share of stock.

(b) $f_u = \max(0, 0) = 0$

$$f_d = \max(0, 3.06) = 3.06$$

$$\phi_u = 0$$

$$\phi_d = -\frac{1}{2}$$

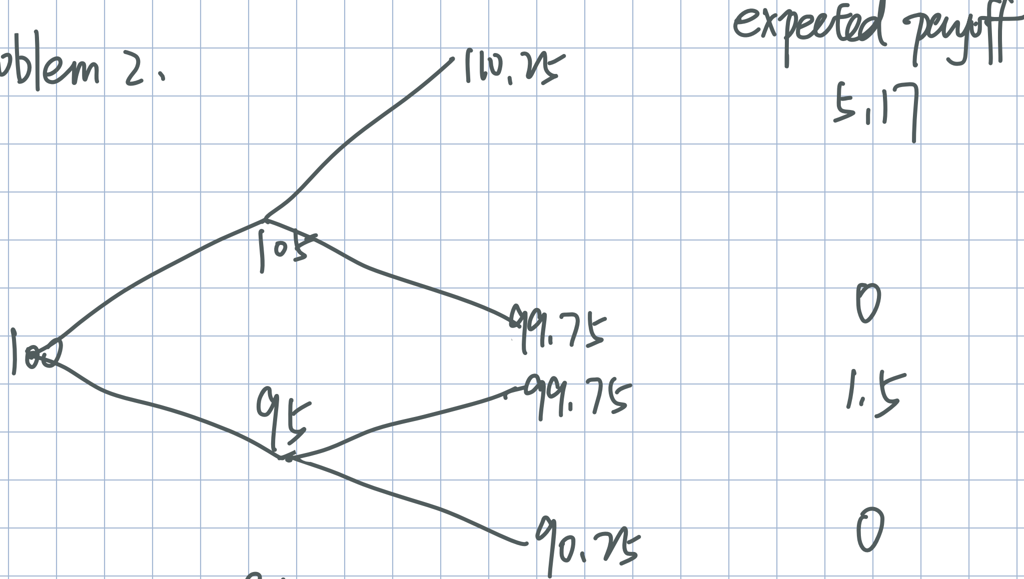
$$f = \max\left(0, 1.03^{-1}(0.65\times 0 + 0.35\times 3.06)\right) = 1.04$$

$$\phi = -0.15$$

strategy:

You can choose to exercise or not if the price goes up at t_1. Others are same as before.

Problem 2.



$$q = \frac{1.02 - 0.95}{1.05 - 0.95} = 0.7$$

$$f_u = 1.02^{-1}(0.7 \times 5.17 + 0.3 \times 0) = 3.55$$

$$f_d = 1.02^{-1}(0.7 \times 1.5 + 0) = 1.03$$

$$\phi_u = \frac{5.17 - 0}{110.25 - 99.75} = 0.49$$

$$\phi_d = \frac{1.5 - 0}{99.75 - 90.25} = 0.16$$

$$f = 1.02^{-1}(0.7 \times 3.55 + 0.3 \times 1.03) = 2.74$$

$$\phi = \frac{3.55 - 1.03}{105 - 95} = 0.25$$

strategy: t_0: buy 0.25 share of stock

t_1 $\nearrow$ buy extra 0.24 share of stock

$\searrow$ sell 0.09 share of stock.

Problem 4:

a) $u(t,x) = E\left[e^{-r(T-t)} f(B_T) \mid B_t = x\right]$

$= E\left[e^{-r(T-t)} f(B_T - B_t + B_t) \mid B_t = x\right]$

$= E\left[e^{-r(T-t)} f(\underbrace{B_T - B_t}_{= Y \sim N(0,\, T-t)} + x)\right]$

$= e^{-r(T-t)} E\left[e^{\lambda(Y+x)}\right]$

$= e^{-r(T-t)} \cdot e^{\lambda x} \; E(e^{\lambda Y})$ (characteristic function of $N(\mu, \sigma^2)$ is $e^{\mu + \frac{1}{2}\sigma^2}$)

$= e^{-r(T-t)} e^{\lambda x} \cdot e^{\frac{1}{2}\lambda^2 (T-t)}$

b) $\partial_t u = e^{\lambda x} \cdot e^{-r(T-t)} \cdot e^{\frac{1}{2}\lambda^2(T-t)} \cdot \left(r - \frac{1}{2}\lambda^2\right)$

$\partial_x u = e^{\lambda x} \cdot e^{-r(T-t)} \cdot e^{\frac{1}{2}\lambda^2(T-t)} \cdot (\lambda)$

$\partial_{xx} u = e^{\lambda x} \cdot e^{-r(T-t)} \cdot e^{\frac{1}{2}\lambda^2(T-t)} \cdot (\lambda^2)$

c) $\partial_t u + \frac{1}{2}\partial_{xx}^2 u = \left(r - \frac{1}{2}\lambda^2\right) u + \frac{1}{2}\lambda^2 \cdot u = ru$

d) $E\left[e^{-r(T-t)} f(B_T) \frac{B_T - B_t}{T-t} \mid B_t = x\right]$

$= e^{-r(T-t)} E\left[f(\underbrace{B_T - B_t}_{= Y \sim N(0,\, T-t)} + x) \cdot \frac{B_T - B_t}{T-t}\right]$

$= e^{-r(T-t)} E\left[e^{\lambda(Y+x)} \cdot \frac{Y}{T-t}\right]$

$= \frac{e^{-r(T-t)}}{T-t} \cdot e^{\lambda x} \; E\left[e^{\lambda Y} \cdot Y\right]$

$= \frac{e^{-r(T-t)} \cdot e^{\lambda x}}{T-t} \cdot \int_{\mathbb{R}} y \cdot e^{\lambda y} \cdot \frac{1}{\sqrt{2\pi(T-t)}} \cdot e^{-\frac{y^2}{2(T-t)}} \, dy$

$$= \frac{e^{-r(T-t)} \cdot e^{\lambda x}}{(T-t)\sqrt{2\pi(T-t)}} \int_R e^{\lambda y} \cdot e^{-\frac{y^2}{2(T-t)}} \cdot d\left(\frac{y^2}{2}\right)$$

$$= \frac{e^{-r(T-t)} \cdot e^{\lambda x}}{(T-t)\sqrt{2\pi(T-t)}} \int_R e^{\lambda y} \, d\left(e^{-\frac{y^2}{2(T-t)}} \cdot (-1)(T-t)\right)$$

$$= \frac{-e^{-r(T-t)} \cdot e^{\lambda x}}{\sqrt{2\pi(T-t)}} \int_R e^{\lambda y} \, d\left(e^{-\frac{y^2}{2(T-t)}}\right)$$

$$= \frac{-e^{-r(T-t)} \cdot e^{\lambda x}}{\sqrt{2\pi(T-t)}} \cdot \left[\underbrace{e^{\lambda y} \cdot e^{-\frac{y^2}{2(T-t)}} \Big|_{-\infty}^{\infty}}_{=0} - \int_R e^{-\frac{y^2}{2(T-t)}} \cdot e^{\lambda y} \cdot \lambda \, dy\right]$$

$$= \frac{\lambda \cdot e^{\lambda x} \cdot e^{-r(T-t)}}{\sqrt{2\pi(T-t)}} \int_R e^{-\frac{y^2 - \lambda y \cdot 2(T-t)}{2(T-t)}} \, dy$$

$$= \frac{\lambda \cdot e^{\lambda x} \cdot e^{-r(T-t)}}{\sqrt{2\pi(T-t)}} \int_R e^{-\frac{(y-\lambda(T-t))^2}{2(T-t)} + \frac{\lambda^2 (T-t)^2}{2(T-t)}} \, dy$$

$$= \frac{\lambda \cdot e^{\lambda x} \cdot e^{-r(T-t)} \cdot e^{\frac{\lambda^2}{2}(T-t)}}{\sqrt{2\pi(T-t)}} \cdot \underbrace{\int_R e^{-\frac{(y-\lambda(T-t))^2}{2(T-t)}} \, dy}_{=\sqrt{2\pi(T-t)}}$$

$$= \lambda \cdot e^{\lambda x} \cdot e^{-r(T-t)} \cdot e^{\frac{\lambda^2}{2}(T-t)}$$

$$= \partial_x u$$

e) $\frac{d}{dx} E\left[e^{-r(T-t)} g(B_T) \mid B_t = x\right]$

$$= \frac{d}{dx} E\left[e^{-r(T-t)} g(\underbrace{B_T - B_t}_{Y \sim N(0,\, T-t)} + x) \right]$$

$$= \frac{d}{dx} \int_R e^{-r(T-t)} g(y+x) \cdot \frac{1}{\sqrt{2\pi(T-t)}} \cdot e^{-\frac{y^2}{2(T-t)}} dy$$

(since g is bounded and measurable, $\frac{d}{dx}$ can be put into the integral) $\frac{d}{dx} g(y+x) = \frac{d}{dy} g(y+x)$

$$= \int_R e^{-r(T-t)} g'(y+x) \cdot \frac{1}{\sqrt{2\pi(T-t)}} \cdot e^{\frac{-y^2}{2(T-t)}} dy$$

(integral by parts)

$$= \int_R e^{-r(T-t)} \cdot \frac{1}{\sqrt{2\pi(T-t)}} \cdot e^{\frac{-y^2}{2(T-t)}} d(g(y+x))$$

$$= e^{-r(T-t)} \left[\underbrace{\frac{1}{\sqrt{2\pi(T-t)}} \cdot e^{\frac{-y^2}{2(T-t)}} \cdot g(y+x) \Big|_{-\infty}^{\infty}}_{=0} - \int_R g(y+x) \frac{1}{\sqrt{2\pi(T-t)}} \cdot e^{\frac{-y^2}{2(T-t)}} \cdot \frac{-y}{T-t} dy \right]$$

(g is bounded)

$$= e^{-r(T-t)} \int_R g(y+x) \cdot \frac{y}{T-t} \cdot \frac{1}{\sqrt{2\pi(T-t)}} \cdot e^{\frac{-y^2}{2(T-t)}} dy$$

$$= E\left[e^{-r(T-t)} g(Y+x) \cdot \frac{Y}{T-t} \right]$$

$$= E\left[e^{-r(T-t)} g(B_T) \frac{B_T - B_t}{T-t} \,\Big|\, B_t = x \right]$$

Problem 3.

(a). $\boxed{S_t = S_0 \exp((r - \sigma^2/2)t + \sigma B_t^{Q})}$

(b). $E^{Q}[e^{-rT} \mathbb{1}_{\{S_T \geq K\}}]$

$= e^{-rT} E^{Q}[\mathbb{1}_{\{S_T \geq K\}}]$

$= e^{-rT} Q[S_T \geq K]$

Since $S_T = S_0 \exp((r - \sigma^2/2)T + \sigma B_T^{Q})$, then:

$Q[S_T \geq K] = Q[B_T^{Q} \geq \frac{\log(K/S_0) - (r - \sigma^2/2)T}{\sigma}]$

$= Q[Z \geq \frac{\log(K/S_0) - (r - \sigma^2/2)T}{\sigma\sqrt{T}}]$

$= Q[Z \leq \frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}]$

where $Z \sim N(0,1)$.

So $Q[S_T \geq K] = \Phi(d_2)$. $d_2 = \frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$

Hence: $\boxed{E^{Q}[e^{-rT} \mathbb{1}_{\{S_T \geq K\}}] = e^{-rT}\Phi(d_2)}$

(c): Let $d_1(K) = \frac{\log(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, $d_2(K) = \frac{\log(S_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$

$E^{Q}[e^{-rT} g(S_T)] = e^{-rT} E^{Q}[S_T \mathbb{1}_{S_T \leq K_1} + \frac{K_1}{K_2 - K_1}(K_2 - S_T)\mathbb{1}_{K_1 < S_T \leq K_2} + (S_T - K_2)\mathbb{1}_{S_T > K_2}]$

Notice that: $e^{-rT} E^{Q}[S_T \mathbb{1}_{S_T \leq K_1}] = S_0 \Phi(-d_1(K_1))$

And $e^{-rT} E^{Q}[(S_T - K_2)\mathbb{1}_{S_T > K_2}] = e^{-rT} E^{Q}[(S_T - K_2)_+]$

$= S_0 \Phi(d_1(K_2)) - K e^{-rT}\Phi(d_2(K_2))$

On the other hand:

$E^{Q}[\mathbb{1}_{K_1 < S_T \leq K_2}] = Q[K_1 < S_T \leq K_2]$

$= Q[K_1 < S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma B_T^{Q}} \leq K_2]$

$= Q[\frac{\log(K_1/S_0) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} < Z \leq \frac{\log(K_2/S_0) - (r - \sigma^2/2)T}{\sigma\sqrt{T}}]$

$Z \sim N(0,1)$

$= \Phi(-d_2(K_2)) - \Phi(-d_2(K_1)) = \Phi(d_2(K_1)) - \Phi(d_2(K_2))$

And $\mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}_{K_1 < S_T \leq K_2}]$

$$= \mathbb{E}^{\mathbb{Q}}[S_T(\mathbb{1}_{S_T \leq K_2} - \mathbb{1}_{S_T \leq K_1})]$$

$$= \mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}_{S_T < K_2}] - \mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}_{S_T < K_1}]$$

$$= e^{rT} S_0 \Phi(-d_1(K_2)) - e^{rT} S_0 \Phi(-d_1(K_1))$$

$$= e^{rT} S_0 \left(\Phi(d_1(K_1)) - \Phi(d_1(K_2)) \right)$$

As a consequence:

$$\boxed{\begin{aligned} \mathbb{E}^{\mathbb{Q}}[e^{-rT} g(S_T)] = {} & S_0 \Phi(-d_1(K_1)) + \frac{K_1 K_2}{K_2 - K_1} e^{-rT} \left(\Phi(d_2(K_1)) - \Phi(d_2(K_2)) \right) \\ & - \frac{K_1}{K_2 - K_1} \left(S_0 \Phi(d_1(K_1)) - S_0 \Phi(d_1(K_2)) \right) \\ & + S_0 \Phi(d_1(K_2)) - K_2 e^{-rT} \Phi(d_2(K_2)) \end{aligned}}$$