MATH4210: Financial Mathematics Tutorial 4

Jiazhi Kang

The Chinese University of Hong Kong

jzkang@math.cuhk.edu.hk

6 February, 2024

Review on Normal r.v.

For X, Y two r.v.s. and $\forall a, b \in \mathbb{R}$, then

$$\mathbb{E}[aX+bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$$

and

$$Var(aX+bY) = a^2 Var(X) + 2ab\,Cov(X,Y) + b^2 Var(Y).$$

Question

Assume a sequence of i.i.d. r.v.s $\{X_i\}_{i=1\dots n}$, $X_1 \sim N(\mu, \sigma^2)$. *Denote by* $Y := \frac{1}{n}\sum_{i=1}^{n} X_i$. *Find* $\mathbb{E}[Y]$ *and* $Var(Y)$

$$\mathbb{E}[Y] = \mathbb{E}\Big[\frac{1}{n}\sum_{i=1}^{n} X_i\Big] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \mu.$$

$$Var(Y) = Var\Big(\frac{1}{n}\sum_{i=1}^{n} X_i\Big) = \frac{1}{n^2} Var\Big(\sum_{i=1}^{n} X_i\Big)$$

(independent)

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)$$

$$= \frac{1}{n}\cdot\sigma^2$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{n\to+\infty} \mathbb{E}[X_1]$$

$n \to +\infty$

$Var(Y) \to 0$

Review on Normal r.v.

To show a r.v. has any specific distribution:

① investigate the characteristic function (or Moment generating function)

$t \in \mathbb{C}$: $\varphi_X(t) = \mathbb{E}[e^{itX}]$ (characteristic func).

$t \in \mathbb{R}$: $M_X(t) = \mathbb{E}[e^{tX}]$. (Moment generating function).

Question

Assume a sequence of independent r.v.s $\{X_i\}_{i=1\ldots n}$, for any $i \in [|1, n|]$, $X_i \sim N(\mu_i, \sigma_i^2)$. Denote by $Y := \sum_{i=1}^{n} X_i$. Show that Y is gaussian. Find the parameters of Y.

Y is normal.

If $X \sim N(\mu, \sigma^2)$. Then $\varphi_X(t) = \mathbb{E}[e^{itX}] = e^{\mu t - \frac{1}{2}\sigma^2 t^2}$

$M_X(t) = \cdots = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

② Identify its cdf / pdf / pmf

$\mathbb{P}[X \leq x] = ?$ $\quad p_X(x) = \cdots = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ (normal).

For $t \in \mathbb{R}$.

$$M_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{t\sum_{i=1}^n X_i}]$$

$$= \mathbb{E}[\prod_{i=1}^n e^{tX_i}]$$

(independent)
$$= \prod_{i=1}^n \underbrace{\mathbb{E}[e^{tX_i}]}_{= M_{X_i}(t)}$$

$$= \prod_{i=1}^n M_{X_i}(t)$$

$$= \prod_{i=1}^n e^{t\mu_i + \frac{1}{2}\sigma_i^2 t^2}$$

$$= e^{t\underline{\sum_{i=1}^n \mu_i} + \frac{1}{2}\underline{\sum_{i=1}^n \sigma_i^2} t^2}$$

Then we conclude that $Y \sim N(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$.

Review on Normal r.v.

Fix $x \in \mathbb{R}_+^*$

$$\mathbb{P}[X \le x] = \mathbb{P}[\underbrace{\log X}_{N(\mu,\sigma^2)} \le \log x] = \int_{-\infty}^{\log x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

"cdf"

Question

Assume X follows log-normal distribution with parameters μ, σ^2. Find the probability density function of X.
Recall that $X \sim LN(\mu, \sigma^2)$ if

$$\ln(X) \sim N(\mu, \sigma^2)$$

Denote the pdf of X at $x \in \mathbb{R}_+^*$ as $p(x)$.

So $\mathbb{P}[X \le x] = \int_0^x p(y)dy.$

Therefore $p(x) = \frac{d}{dx}(\mathbb{P}[X \le x]) = \frac{d}{dx}(\log x) \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \boxed{\frac{1}{x\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$

$F(x) = \int_0^{\varphi(x)} f(y)dy$, $F'(x) = \varphi'(x) \cdot f(\varphi(x))$.

Review on Normal r.v.

$M_X(t) = \mathbb{E}[e^{tX}] \in C^\infty(\mathbb{R}, \mathbb{R}^+)$

$M_X^{(n)}(0) = \mathbb{E}[X^n]$

Question

Assume $X \sim N(0,1)$. For $n \in \mathbb{N}^*$, find $\mathbb{E}[X^n]$. n-th moment of X.

Remark

If $Y \sim N(\mu, \sigma^2)$, we can write $\mathbb{E}[Y^n]$ explicitly by considering

$$Y = \mu + \sigma Z, \quad Z \sim N(0,1).$$

① Use Moment generating function

② Compute directly.

① $M_X(t) = \mathbb{E}[e^{tX}] = e^{t\mu + \frac{1}{2}\sigma^2 t^2}$ $\quad X \sim N(\boxed{0}, 1)$

$= e^{\frac{1}{2}t^2}$

$M_X'(t) = t\, e^{\frac{1}{2}t^2}$

$= t \cdot M_X(t)$

$M_X''(t) = M_X(t) + t \cdot M_X'(t)$

$M_X^{(3)}(t) = M_X'(t) + M_X'(t) + t M_X''(t)$

$= 2M_X'(t) + t M_X''(t)$

$M_X^{(4)}(t) = 3M_X''(t) + t M_X^{(3)}(t)$

$t = 0$. then

$M_X'(0) = 0$

$M_X''(0) = M_X(0) = 1$.

$\vdots$

$M_X^{(n)}(t) = (n-1) M_X^{(n-2)}(t) + t M_X^{(n-1)}(t)$

Goal: $M_X^{(n)}(0) = (n-1) M_X^{(n-2)}(0) \iff \mathbb{E}[X^n] = (n-1)\mathbb{E}[X^{n-2}]$

1st case: n odd, then $\mathbb{E}[X^n] = 0$

since it relies on $\mathbb{E}[X^1] = \mu = 0$

2nd case: n even

Then $\mathbb{E}[X^n] = (n-1)\mathbb{E}[X^{n-2}] = (n-1)\cdot(n-3)\cdot\mathbb{E}[X^{n-4}] \cdots$

$= 1 \times 3 \times \cdots \times (n-1)$

For all $k \in \mathbb{N}$:

$$\begin{cases} \mathbb{E}[X^{2k+1}] = 0 \\ \mathbb{E}[X^{2k}] = \dfrac{(2k)!}{2^k k!} \end{cases}$$

② $\mathbb{E}[X^n] = \int_{\mathbb{R}} x^n \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^{n-1} \cdot \underbrace{\left(x e^{-\frac{x^2}{2}}\right)} dx$.

$= \cdots$ primitive: $-e^{-\frac{x^2}{2}}$

$= (n-1)\mathbb{E}[X^{n-2}]$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. X and $\{X_n\}$ are $\mathbb{R}$ valued (sequence of) r.v.s.

Definition (Convergence almost surely)

Denote by $X_n \to X$ a.s. (almost surely) if

convergence almost everywhere

$$\mathbb{P}[\{\omega \in \Omega : \boxed{\lim_{n\to\infty} X_n(\omega)} = X(\omega)\}] = 1$$

stronger

Definition (Convergence in Probability)

Denote by $X_n \to X$ in probability if for any $\rho > 0$

convergence in measure

$$\boxed{\lim_{n\to\infty}} \mathbb{P}[\{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geq \rho\}] = 0$$

CV a.e. $\Rightarrow$ CV. in measure

Question

Let X and $\{X_n\}$ be $\mathbb{R}$ valued (sequence of) r.v.s. Assume $X_n \to X$ a.s.. Show that $X_n \to X$ in probability.

Goal: $\lim_{n\to+\infty} \mathbb{P}[\{\omega\in\Omega : |X_n(\omega) - X(\omega)| \geq \rho\}] = 0$ for any $\rho > 0$.

$\mathbb{P}[\{|X_n - X| \geq \rho\}] \to 0$

$A_n = \bigcup_{m\geq n} \{|X_m - X| \geq \rho\}. \Rightarrow \{|X_n - X| \geq \rho\} \subseteq A_n.$

$\Downarrow$ by monotonicity of probability measure.

$\mathbb{P}[\{|X_n - X| \geq \rho\}] \leq \mathbb{P}(A_n)$

If $\lim_{n\to+\infty} \mathbb{P}(A_n)=0$ then we're done.

A_n is decreasing $\Rightarrow \lim_{n\to+\infty} \mathbb{P}(A_n) = \mathbb{P}(\bigcap_{n\geq 1} A_n)$

(increasing $\Rightarrow \lim \mathbb{P}(A_n) = \mathbb{P}(\bigcup A_n)$.)

$$\lim_{n\to+\infty} \mathbb{P}[\{|X_n - X| \geq \rho\}] \leq \lim_{n\to+\infty} \mathbb{P}(A_n)$$

$$= \mathbb{P}(\bigcap_{n\geq 1} A_n)$$

Goal: $\mathbb{P}(\bigcap_{n\geq 1} A_n) = 0$ $\qquad \mathbb{P}(\Omega) = 1$

$$\mathbb{P}\Big(\big(\bigcap_{n\geq 1} A_n\big) \cdot \underset{\parallel}{\cup} B\Big) \leq \mathbb{P}(\Omega) = 1.$$

what if $\mathbb{P}(\bigcap_{n\geq 1} A_n) + \underset{\underset{1}{\parallel}}{\mathbb{P}(B)}$

Let $B = \{\lim_{n\to+\infty} X_n = X\}$, since $X_n \xrightarrow{a.s.} X$, then $\mathbb{P}(B) = 1$

Show: $\mathbb{P}\Big(\big(\bigcap_{n\geq 1} A_n\big) \cup B\Big) = \mathbb{P}(\bigcap_{n\geq 1} A_n) + \mathbb{P}(B)$

So it suffices to show $\big(\bigcap_{n\geq 1} A_n\big) \cap B = \emptyset$

$\forall \varepsilon > 0, \exists N \geq 0 \ldots\ldots$

Take $\omega \in B$, since $X_n \to X$ a.s. then there exists some $N \geq 0$ s.t. $|X_n(\omega) - X(\omega)| < \rho$ for all $n \geq N$

Therefore $\omega \notin A_n$ for all $n \geq N$

Then $\omega \notin \bigcap_{n\geq 1} A_n$.

Hence: $B \cap (\bigcap_{n\geq 1} A_n) = \emptyset$

So $\lim_{n\to+\infty} \mathbb{P}[\{|X_n - X| \geq \rho\}] \leq \lim_{n\to+\infty} \mathbb{P}(A_n)$

$$= \mathbb{P}(\bigcap_{n\geq 1} A_n)$$

But $\mathbb{P}(B \cup (\bigcap_{n\geq 1} A_n)) = \underset{1}{\mathbb{P}(B)} + \underbrace{\mathbb{P}(\bigcap_{n\geq 1} A_n)}_{=0} \leq \mathbb{P}(\Omega) = 1 \Rightarrow \mathbb{P}(\bigcap_{n\geq 1} A_n) = 0$