MATH4210: Financial Mathematics

V. General Theory on Derivatives

Financial Assets

The basic types of financial assets are *debt*, *equity*, and *derivatives*.

- *Debt* instruments are issued by anyone who borrows money - firms, governments, and households. The assets traded in debt markets, therefore, include corporate bonds, government bonds, residential and commercial mortgages, and consumer loans. These debt instruments are also called *fixed-income* instruments because they promise to pay fixed amout of cash in the future.
- *Equity* is the claim of the owners of a firm. Equity securities issued by corporations are called common stocks or shares. They are bought and sold in the stock markets.
- *Derivatives* are financial instruments that derive their value from the prices of one or more other assets such as equity securities, fixed-income securities, foreign currencies, or commodities. Their main function is to serve as tools for managing exposures to the risks associated with the underlying assets.

Derivatives

We will consider the derivative product whose payoff at *maturity time T* is given by

$$X := g(S_T), \quad \text{or more generally} \quad X := g((S_t)_{t\in[0,T]}),$$

where g is a function, S denotes the underlying (risky asset price) process.

We will consider the case where *S is a stochastic process* with given distribution.

Example

- Forward: $X = S_T - F(0,T)$.
- Call option $X = (S_T - K)_+$.
- Put option $X = (K - S_T)_+$.
- Digital option $X = 1_{\{S_T > K\}}$.
- Asian option $X = (A_T - K)_+$ with $A_T := \int_0^T S_t dt$.

Underlying process: discrete time setting

CRR (Cox, Ross and Rubinstein) model, also called the binomial tree model: let $p \in (0,1)$, $0 < d < 1 + r < u$ where r denotes the interest rate.

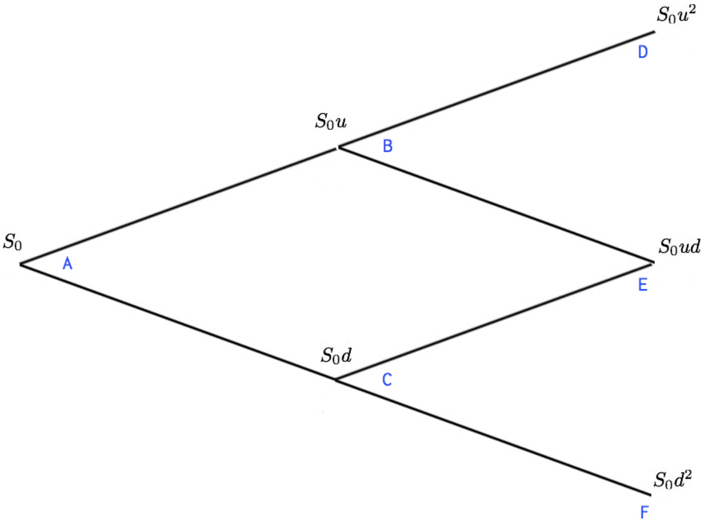
The dynamic of S on the discrete time grid $0 = t_0 < t_1 < \cdots < t_n$ is given by

$$\mathbb{P}\big[S_{t_{k+1}} = uS_{t_k}\big] = p, \quad \mathbb{P}\big[S_{t_{k+1}} = dS_{t_k}\big] = 1 - p.$$

Consequently,

$$\mathbb{P}\big[S_{t_n} = S_0 u^k d^{n-k}\big] = C_n^k \; p^k (1-p)^{n-k}.$$

Underlying process: discrete time setting



Underlying process: continuous time setting

We say $B = (B_t)_{t \geq 0}$ is a standard *Brownian motion* if

- $B_0 = 0$,
- the map $t \mapsto B_t$ is continuous on $[0, \infty)$,
- $B_t - B_s \sim N(0, t - s)$, for all $0 \leq s \leq t$,
- $B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}$ are independent for all $0 \leq t_0 \leq \cdots \leq t_n$.

The *Black-Scholes* model is given by

$$S_t = S_0 \exp\big((r - \sigma^2/2)t + \sigma B_t\big),$$

where S_0 is the initial stock price, r denotes the (continuously compounded) interest rate, σ is called the volatility parameter.

Arbitrage Opportunity

A portfolio (trading strategy) is a combination of different financial assets (at different time).

A portfolio is said to be self-financing if one does not add or withdraw money from the portfolio at an intermediate time $t \in (0, T)$. Without further precision, a portfolio is always assumed to be self-financing.

Arbitrage opportunity exists if the value of a portfolio is 0 at initial time, and is non-negative in all scenarios and positive in some scenario at the end. That is

$$\Pi_0 = 0, \quad \mathbf{P}[\Pi_T \geq 0] = 1, \quad \mathbf{P}[\Pi_T > 0] > 0,$$

where Π_t is the value of a portfolio at time t.

No Arbitrage Condition

We always assume the No Arbitrage condition, i.e., there is no (self-financing) portfolio providing an arbitrage opportunity.

Theorem 1 ((Pricing by replication))

Assume that a derivative has payoff X (which could be a random variable) at time T, and there is a portfolio Π such that

$$\Pi_T = X, \text{ almost surely.}$$

Then the price of the derivative at time t is given by

$$\Pi_t.$$

Forward Contract

- A Forward contract is a commitment to purchase at a future date a given amount of a commodity or an asset at a price agreed on today (forward price).
- Another way to look at a forward contract is to say that the parties agree to lock the price of the future transaction.
- **Examples**:
  - A forward contract to purchase a non-dividend paying stock with the forward price 100 per share and the maturity $T = 6$ months.
  - A forward contract to purchase gold with the pre-determined price with maturity 1 year.
- The initial price of the underlying asset is called the "spot price". The initial value of the forward contract is always 0.

Forward price

- Let the forward contract is signed at time 0, the delivery time (maturity) is T and the forward price (of the underlying) is denoted by $F(0,T)$. Then the payoff of the forward contract at the maturity is

$$S_T - F(0,T).$$

- Suppose that we have continuous compounding interest rate r in the market. For a stock paying no dividends the forward price is

$$F(0,T) = S_0 e^{rT}.$$

It the contract is initiated at time $t \leq T$, we denote the forward price by $F(t,T)$ and

$$F(t,T) = S_t e^{r(T-t)}.$$

Value of the forward contract at time t

- Every forward contract has value zero when initiated. As time goes by, the price of the underlying asset may change. Along with it, the value of the forward contract, denoted by $V(t)$, will vary and will no longer be zero.
- Suppose that the forward price $F(t,T)$ for a forward contract initiated at time t is higher than $F(0,T)$. This is good news for an investor with a long forward position initiated at time 0. At time T, such an investor will gain $F(t,T) - F(0,T)$. To find the value of the original forward position at time t, all we have to do is to discount this gain back to time t.

Value of the forward contract at time t

For any $0 \leq t \leq T$, the time t value of a long forward contract with the forward price $F(0,T)$ satisfies

$$V(t) = [F(t,T) - F(0,T)]e^{-r(T-t)}.$$

Forward price in case with dividend

- Case 1: A deterministic amount D of dividend is delivered at time $t_D \in (0,T)$ if one holds one stock:

$$F(0,T) = S_0 e^{rT} - De^{r(T-t_D)}.$$

- Case 2: A proportional dividend $(d \times S_{t_D-})$ at time time $t_D \in (0,T)$ if one holds one stock, where $d \in (0,1)$:

$$F(0,T) = (1-d)S_0 e^{rT}.$$

Futures Contract

- One of the two parties to a forward contract will be losing money and there is always a risk of default by the party suffering a loss. Futures contract are designed to eliminate such risk.
- A futures contract: it is similar to a forward contract except that
  - futures are traded on organized exchanges
  - the fluctuations in the price of the underlying instrument are settled by daily payment between the parties
- This process of daily adjustment is known as "marking to market" and is meant to reduce the risk of default.
-
  - At time $t_0 = 0$, one initiates a future contract, by fixing $f(t_0, T)$, which costs nothing.
  - Then at each time step $t_k \in \{t_1, t_2, \ldots, t_n = T\}$, the holder of a long futures contract will receive the amount

$$f(t_k, T) - f(t_{k-1}, T)$$

if positive, or will have to pay if it is negative.

Futures Contract

- Two conditions are imposed:
  1. The futures price (of the underlying) at delivery/maturity time T is
  $$f(T,T) = S_T.$$
  2. At each time step t_k, one computes the value
  $$f(t_k,T) \text{ (marking to market)},$$
  to make the value of a futures contract be zero after the payment $f(t_k,T) - f(t_{k-1},T)$.
- The second condition means that, in particular, it costs nothing to close or open a futures position at any time step between 0 and T.

Futures Contract

Theorem 2

Assume that the interest rate is a constant r, then

$$f(t_k, T) = F(t_k, T) = e^{r(T-t_k)} S_{t_k}.$$

Vanilla Option

A vanilla European call (or put) option gives the holder of it the right to buy (or sell) one unit of underlying asset S_T from the writer at the strike price K at maturity T.

$$\text{Payoff is } (S_T - K)_+ \text{ (0r } (K - S_T)_+) \text{ at time } T.$$

A vanilla American call (or put) option gives the holder of it the right to buy (or sell) one unit of underlying asset S_t from the writer at the strike price K at any time $t \in [0, T]$.

Payoff is $(S_t - K)_+$ (0r $(K - S_t)_+$) at time t if it is exercised at time t.

Other Options

European and American call and put options form a small section of the available derivative products. They are called vanilla options because they are popular and simple.

There are however other more complicated options, including the so-called exotic and path-dependent options. These options have values which depend on the history of an asset price, not just on its value on exercise. An example is an option to purchase an asset for the arithmetic average value of that asset over the month before expiry. An investor might want such an option in order to hedge sales of a commodity, say, which occur continually throughout this month. There are also options which depend on the geometric average of the asset price, the maximum or the minimum of the asset price.

Other Options

Examples

- Digital option: an option that at maturity pays a fixed amount if in-the-money, zero otherwise. A digital option is also called a binary option.
- Barrier options: the option which can either come into existence or become worthless if the underlying asset reaches some prescribed value before expiry.
- Asian options: price depends on some form of average. If S_T is replaced by the average, it is called Asian price option. If K is replaced by the average, it is called Asian strike option.
- Lookback options: price depends on the maximum or minimum of the underlying asset price.

No Arbitrage Condition

Comparison of prices

Proposition 3.1

Assume that there are two derivative options with respectively payoff X_1 and X_2 at the same maturity time T, satisfying

$$X_1 \leq X_2, \text{ a.s.}$$

Then the price P_1 and P_2 of the two options at any time t satisfy

$$P_1(t) \leq P_2(t).$$

Remark 1

Remember the slogan: "buy low, sell high" and it will be easier for you to construct a portfolio.

No Arbitrage Condition

Comparison of prices

Proposition 3.2

Assume that there are two derivative options with respectively payoff X_1 and X_2 at the same maturity time T, satisfying

$$X_1 \leq X_2, \text{ a.s.}$$

Moreover, $\mathbb{P}\big[X_1 < X_2\big] > 0$.
Then the price P_1 and P_2 of the two options at any time t satisfy

$$P_1(t) < P_2(t).$$

Vanilla European Option

Assumption 3.1

From now on, we assume the underlying asset price is positive and can be arbitrarily small and large at any time in the future. Moreover, the underlying asset pays no dividend during the life time of the option unless otherwise specified.

Proposition 3.3

If the price of the underlying asset of a vanilla European call option could exceed the strike price at maturity, then the option price is positive before maturity.

Vanilla European Option

Vanilla European Option

Proposition 3.4

At any time before or at maturity, the underlying stock price is always higher than the vanilla European call option price.

Proposition 3.5

For the vanilla European call option with maturity T and strike K, its price $C_E(t,T)$ at time t satisfies

$$C_E(t,T) > S_t - Ke^{-r(T-t)}.$$

Vanilla European Option

Vanilla European Option

Proposition 3.6

Suppose two vanilla European options are identical except for the strike prices $0 < K_1 < K_2$, *then*

$$0 < C_E(t, K_1) - C_E(t, K_2) < (K_2 - K_1)e^{-r(T-t)},$$
$$0 < P_E(t, K_2) - P_E(t, K_1) < (K_2 - K_1)e^{-r(T-t)},$$

at any time t *before maturity* T.

Vanilla European Option

Vanilla European Option

Vanilla European Option

Theorem 3.1 (Put-Call Parity Relation without Dividend)

If the underlying asset pays no dividend, then

$$C_E(t,K) - P_E(t,K) = S_t - Ke^{-r(T-t)}.$$

Proof.

Portfolio 1. At time t, long 1 call option and short 1 put option, so that

$$\Pi_1(t) = C_E(t,K) - P_E(t,K)$$

$$\text{and} \quad \Pi_1(T) = (S_T - K)_+ - (K - S_T)_+ = S_T - K.$$

Portfolio 2. At time t, borrow cash $Ke^{-r(T-t)}$ from bank, and long 1 stock, so that

$$\Pi_2(t) = S_t - Ke^{-r(T-t)}$$

$$\text{and} \quad \Pi_2(T) = S_T - K.$$

Vanilla European Option

By replication argument, $\Pi_1(t) = \Pi_2(t)$.

Vanilla European Option

Theorem 3.2 (Put-Call Parity Relation with Dividend)

Assume that the value of the dividends of the stock paid during $[t, T]$ is a deterministic constant D at time $\tau_D \in (t, T]$. Then

$$C_E(t, K) - P_E(t, K) = S_t - Ke^{-r(T-t)} - De^{-r(\tau_D - t)}.$$

Remark 2

Left as an exercise.

Vanilla European Option

Proposition 3.7

Two vanilla call options are identical except for the maturity dates $T_1 < T_2$. Then $C_E(t, T_1) < C_E(t, T_2)$ at any time $t \leqslant T_1$.

Proof.

Suppose $C_E(t, T_1) \geqslant C_E(t, T_2)$. Make a portfolio: long a call with maturity T_2 and short a call with maturity T_1 at time t. Then the value of the portfolio at time t is

$$\Pi_t = C_E(t, T_2) - C_E(t, T_1) \leqslant 0.$$

At time T_1, the value of the portfolio is

$$\Pi_T = C_E(T_1, T_2) - (S_{T_1} - K)_+ > 0,$$

where the last inequality follows by Proposition 3.5. This is an arbitrage. □

Vanilla European Option

Proposition 3.8

Two vanilla put options are identical except for the maturity dates $T_1 < T_2$. If the interest rate is zero between T_1 and T_2, then $P_E(t, T_1) < P_E(t, T_2)$ at any time $t \leqslant T_1$.

Proof.

Left as an exercise. □

Remark 3

Exercise: What happens if the underlying asset pays dividend?

Remark 4

Exercise: What happens if the interest rate is not always zero?

Portfolios of European Options

- A **Spread** is a transaction in which an investor simultaneously buys one option and sells another option, both on the same underlying asset, but with different terms (strike price and/or expiration time).
- In a **bull spread**, the investor buys a call at a certain strike price K_1 and sells another call at a higher strike price K_2, with the same expiration time.
- In a **bear spread**, the investor buys a call at a certain strike price K_1 and sells another call at a lower strike price K_2, with the same expiration time.
- A **butterfly spread** is a combination of a bull spread and a bear spread. A call butterfly spread consists of buying a call at strike price K_1, selling two calls at strike price $K_2 > K_1$ and buying another call at strike price $K_3 > K_2$. All calls have the same expiration date.

Convexity of Call option prices

Example: Suppose that we have 3 European call options with the same maturity T in the financial market:

Type	Strike Price	Price at time 0
Call	90	10
Call	100	9
Call	110	7

Suppose that the continuous compounding interest rate $r = 0.05$ in the market and the maturity time $T = 1$. Can you construct an arbitrage portfolio with the above options.

Examples of Spread

Example: You look up the prices of European calls today on a particular stock and discover the following prices for call options:

Strike Price	Price
195	47.80
200	45.60
205	44.20

A friend has confidence that this stock will do well in the future. This friend offers to bet any amount of money you like (a constant that you two decide today) that the stock will be worth at least 200 on the expiry date in October, 2013 (which we suppose is exactly one year from today). Can you combine a bet with this friend with portfolios of call options and a riskfree asset (at interest rate 0.05 per year with annual compounding) to make an arbitrage profit with zero initial wealth? If yes, be specific of your strategies and profits.

Examples of Spread

Example: Suppose that we have the following 4 European call and put options with the same maturity T in the financial market:

Type	Strike Price	Price
Call	100	45
Call	110	40
Put	100	36
Put	110	42

Suppose that the continuous compounding interest rate $r = 0.05$ in the market and the maturity time $T = 1$. Can you choose a portfolio using some of the options from the table and the Bank account to find an Arbitrage profit ? If yes, be specific of your arbitrage portfolio. If no, prove your argument.

Vanilla American Option

Call and put options only form a small section of the available derivative products. Other than European options that we have discussed so far, most options nowadays are what are called American options. (Recall that the European/American classification has nothing to do with the continent of origin but refers to a technicality in the option contract.)

An American option is one that may be exercised at any time prior to expiry by its holder. A question that the holder of an American option must determine is when to exercise the option. We will see that the best time to exercise is not subjective, but that it can be determined in a natural and systematic way.

Vanilla American Option

Suppose that, at some time t before maturity, we decide to exercise it, then we must have

$$S_t - K > 0.$$

Otherwise, when exercising it, we would get $S_t - K$ dollars which is non-positive. That means either we get 0 dollars and lose the option, or we have to pay $|S_t - K|$ to the option writer. In either case, we lose money in the deal. Such action is worse than holding the option. That contradicts the idea of "exercising it in an optimal/raitional way".

Vanilla American Option

Suppose that, at some time t before maturity, we decide to exercise it, then we must have

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On the other hand, if we do not exercise it and hold the option till the contract expiration, then the same consideration given to European call can be applied to this case and we get

$$C_A(T) = (S_T - K)^+.$$

Vanilla American Option

Proposition 3.9

At maturity date, one has

$$C_A(T) = C_E(T) = (S_T - K)_+,$$
$$P_A(T) = P_E(T) = (K - S_T)_+.$$

Vanilla American Option

Since the American option gives its holder more rights than that of European option holder, via the right of early exercise, potentially it has a higher value.

Proposition 3.10

At any time t before or at the maturity date, one has

$$C_A(t) \geqslant C_E(t),$$
$$P_A(t) \geqslant P_E(t).$$

Vanilla American Option

Vanilla American Option

Proof.

Suppose $C_A(t) < C_E(t)$ at some time t before maturity. Make a portfolio at time t: long one the American call option and short one the European call option, and deposit an amount of money $x = C_E(t) - C_A(t) > 0$ in the bank. The value of this portfolio at time t is

$$\Pi_t = 0.$$

Since we are the holder of the American call, we have the right to decide when to exercise it (earlier or at the maturity date T). Let us choose to hold it till T. The value of this portfolio at time T is

$$\begin{aligned} \Pi_T &= C_A(T) - C_E(T) + xe^{r(T-t)} \\ &= (S_T - K)^+ - (S_T - K)^+ + xe^{r(T-t)} > 0. \end{aligned}$$

There is an arbitrage opportunity. □

Vanilla American Option

Proposition 3.11

For the call option, at any time t before maturity T, one has $C_A(t) > S_t - Ke^{-r(T-t)}$.

Proof.

(The same argument as in Proposition 3.5). Suppose not. Make a portfolio at time t: short one share of stock, long one American call option and deposit cash $Ke^{-r(T-t)}$ in bank. Its value at time t is

$$\Pi_t = -S_t + C_A(t) + Ke^{-r(T-t)} \leqslant 0.$$

We choose not to exercise before maturity, then its value will be

$$\Pi_T = -S_T + (S_T - K)^+ + K = \max\{K - S_T, 0\} \geqslant 0 \geqslant \Pi_t$$

at maturity T. Moreover, $\mathbf{P}(\Pi_T > 0) = \mathbf{P}(S_T < K) > 0$. So there is an arbitrage opportunity. □

Vanilla American Option

Vanilla American Option

Proposition 3.12

We have $S_t - Ke^{-r(T-t)} < C_A(t) < S_t$ *at any time* t *before maturity* T.

Remark 5

The proof is left as exercise.

Vanilla American Option

Theorem 3.3 (Merton's Theorem)

If the underlying asset pays no dividend, then the vanilla European and American call options have the same value, $C_E(t) = C_A(t)$.

Proof.

If the holder choose to exercise the American call option at some time t before maturity, then the net payoff is $S_t - K$ (intrinsic value). However, since $C_A(t) > S_t - Ke^{-r(T-t)} \geqslant S_t - K$, it has better payoff if he chooses to sell the option. Therefore, it is never optimal to exercise early. So there is no difference between the vanilla European and American call options. □

Vanilla American Option

Vanilla American Option

Proposition 3.13

Show that $P_A(t) < K$ *at any time* t.

Proof.

Since $P_A(T) = (K - S_T)^+$, $P_A(T) < K$. Suppose $P_A(t) \geqslant K$ at some time t before maturity. Make a portfolio at time t: short one American put option and long cash K. The value of this portfolio at time t is

$$\Pi_t = K - P_A(t) \leqslant 0.$$

If the option is exercised at some time $\tau \leqslant T$, then the portfolio at time τ will change to long one share stock, and its value at maturity T will be

$$\Pi_T = S_T > 0 \geqslant \Pi_t.$$

There is an arbitrage opportunity. □

Vanilla American Option

Proof.

If the option is never exercised, then the value of the portfolio at maturity T will be

$$\Pi_T = K > 0 \geqslant \Pi_t.$$

There is an arbitrage opportunity as well. □

Remark 6

Are the vanilla European and American put options the same?

Vanilla American Option

Proposition 3.14

Show that

$$S_t - K \leqslant C_A(t) - P_A(t) \leqslant S_t - Ke^{-r(T-t)}$$

at any time t*.*

Proof. Part 1.

Observe that $C_A(t) = C_E(t)$ and $P_A(t) \geqslant P_E(t)$ always. Thus, it is trivial to see

$$C_A(t) - P_A(t) \leqslant C_E(t) - P_E(t) = S_t - Ke^{-r(T-t)},$$

by making use of the put-call parity for European options. □

Vanilla American Option

Vanilla American Option

Proof. Part 2.

Suppose $S_t - K > C_A(t) - P_A(t)$ at some time t. Make a portfolio at time t: long one call option and deposit K in bank, short one share of stock and one put option. Then

$$\Pi_t = C_A(t) + K - S_t - P_A(t) < 0.$$

If the put option is exercised at some time $\tau \in [t, T]$, then the portfolio will change to long a call and deposit $Ke^{r(\tau-t)} - K$ in bank. Then

$$\Pi_T = (S_T - K)^+ + Ke^{r(T-t)} - Ke^{r(T-\tau)} \geqslant 0.$$

If the put option is never exercised, then

$$\Pi_T = (S_T - K)^+ + Ke^{r(T-t)} - S_T \geqslant 0.$$

There is an arbitrage opportunity. □

Vanilla American Option

Proposition 3.15

Suppose two vanilla American options are identical except for the strike prices $0 < K_1 < K_2$. Then, at any time t before maturity T,

$$0 < C_A(t, K_1) - C_A(t, K_2) < K_2 - K_1,$$
$$0 < P_A(t, K_2) - P_A(t, K_1) < K_2 - K_1.$$

Proof.

Because the vanilla European call option and American call option are the same,

$$0 < C_A(t, K_1) - C_A(t, K_2) = C_E(t, K_1) - C_E(t, K_2) < (K_2 - K_1)e^{-r(T-t)} \leqslant K_2 - K_1.$$

The second one is left as an exercise. □

Vanilla American Option