MATH4210: Financial Mathematics

IV: Continuous Time Market,
Part B: the replication approach

Black-Scholes Model

Assumptions of Black-Scholes model

(1) Stock price $(S_t)_{0 \le t \le T}$ follows the Black-Scholes model:

$$S_t = S_0 \exp\left((\mu - \sigma^2/2)t + \sigma B_t\right),$$

or equivalently,

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$

(2) Risk-free interest rate r is a constant.

(3) Moreover,

- Short selling is allowed.
- No transaction fees.
- All securities are perfectly divisible.
- No dividends during the lifetime of the derivatives.
- Security trading is continuous.
- No arbitrage opportunities.

Dynamic trading

Dynamic trading: let $t_k := k\Delta t$, risky asset price $(S_{t_k})_{k\geq 0}$, interest rate $r \geq 0$.

Discrete time dynamic trading between t_k and t_{k+1}:

$$\begin{aligned}\Pi_{t_{k+1}} &= \phi_{t_k} S_{t_{k+1}} + \left(\Pi_{t_k} - \phi_{t_k} S_{t_k}\right)(1 + r\Delta t) \\ &= \Pi_{t_k} + \left(\Pi_{t_k} - \phi_{t_k} S_{t_k}\right) r\Delta t + \phi_{t_k}\left(S_{t_{k+1}} - S_{t_k}\right).\end{aligned}$$

Then

$$\Pi_{t_n} = \Pi_0 + \sum_{k=0}^{n-1} \left(\Pi_{t_k} - \phi_{t_k} S_{t_k}\right) r\Delta t + \sum_{k=0}^{n-1} \phi_{t_k}\left(S_{t_{k+1}} - S_{t_k}\right).$$

The continuous time limit:

$$\begin{aligned}\Pi_T &= \Pi_0 + \int_0^T (\Pi_t - \phi_t S_t) r dt + \int_0^T \phi_t dS_t \\ &= \Pi_0 + \int_0^T (\Pi_t - \phi_t S_t) r dt + \int_0^T \phi_t \mu S_t dt + \int_0^T \phi_t \sigma S_t dB_t.\end{aligned}$$

Dynamic trading, discounted value

Let $t_k := k\Delta t$, risky asset price $(S_{t_k})_{k\geq 0}$, interest rate $r \geq 0$. We consider the discounted value:

$$\widetilde{S}_{t_k} := S_{t_k}(1+r\Delta t)^{-k}, \quad \text{and } \widetilde{\Pi}_{t_k} := \Pi_{t_k}(1+r\Delta t)^{-k}.$$

Then

$$\widetilde{\Pi}_{t_{k+1}} = \Pi_{t_k} + \phi_{t_k}\big(\widetilde{S}_{t_{k+1}} - \widetilde{S}_{t_k}\big),$$

so that

$$\widetilde{\Pi}_{t_n} = \Pi_0 + \sum_{k=0}^{n-1} \phi_{t_k}\big(\widetilde{S}_{t_{k+1}} - \widetilde{S}_{t_k}\big).$$

The continuous time limit:

$$\widetilde{\Pi}_t := e^{-rt}\Pi_t, \quad \text{and } \widetilde{S}_t := e^{-rt}S_t = S_0 \exp\big((\mu - r - \sigma^2/2)t + \sigma B_t\big),$$

and

$$\widetilde{\Pi}_T = \Pi_0 + \int_0^T \phi_t d\widetilde{S}_t = \Pi_0 + \int_0^T \phi_t(\mu - r)\widetilde{S}_t dt + \int_0^T \phi_t \sigma \widetilde{S}_t dB_t.$$

Dynamic trading strategy

We say a portfolio $(\Pi_t)_{t\in[0,T]}$ is *self-financing* if

$$\begin{aligned} d\Pi_t &= (\Pi_t - \phi_t S_t) r\,dt + \phi_t\,dS_t, \\ \Leftrightarrow \quad \Pi_t &= \Pi_0 + \int_0^t (\Pi_s - \phi_s S_s) r\,ds + \int_0^t \phi_s\,dS_s. \end{aligned}$$

where Π_t denotes the total wealth of the portfolio, ϕ_t denotes the number of the stocks in the portfolio, $\Pi_t - \phi_t S_t$ denotes the wealth invested in the risk-free asset.

Or equivalently, $(\Pi_t)_{t\in[0,T]}$ is *self-financing* if

$$d\widetilde{\Pi}_t = \phi_t\,d\widetilde{S}_t \quad \Leftrightarrow \quad \widetilde{\Pi}_t = \Pi_0 + \int_0^t \phi_s\,d\widetilde{S}_s.$$

Option pricing by replication

Let us consider the European call option with payoff $g(S_T)$, if there is a self-financing portfolio Π such that

$$\Pi_T = g(S_T) \text{ (or equivalently } \widetilde{\Pi}_T = e^{-rT} g(S_T)\text{)},$$

then the option price at time t is given by

$$\Pi_t.$$

Option price at initial time 0 is Π_0.

Black-Scholes Formula

Notice that

$$\widetilde{S}_t = e^{-rt} S_t \implies d\widetilde{S}_t = -re^{-rt} S_t dt + e^{-rt} dS_t.$$

Let $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a smooth function, and

$$\widetilde{V}_t := e^{-rt} u(t, S_t) = e^{-rt} u\big(t, S_0 \exp\left((\mu - \sigma^2/2)t + \sigma B_t\right)\big).$$

Then by Itô's formula,

$$\begin{aligned}
\mathrm{d}\widetilde{V}_t &= \mathrm{d}\Big(e^{-rt} u\big(t, S_0 \exp\left((\mu - \sigma^2/2)t + \sigma B_t\right)\big)\Big) \\
&= e^{-rt}\Big(\partial_t u(t,S_t) + \mu S_t \partial_x u(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_{xx}^2 u(t,S_t) - ru(t,S_t)\Big)\,\mathrm{d}t \\
&\quad + \partial_x u(t,S_t) e^{-rt} \sigma S_t \,\mathrm{d}B_t \\
&= e^{-rt}\Big(\partial_t u(t,S_t) + r S_t \partial_x u(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_{xx}^2 u(t,S_t) - ru(t,S_t)\Big)\,\mathrm{d}t \\
&\quad + \partial_x u(t,S_t)\,\mathrm{d}\widetilde{S}_t.
\end{aligned}$$

Black-Scholes Formula: Delta hedging

Let $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfy

$$\partial_t u(t,x) \;+\; rx\partial_x u(t,x) \;+\; \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} u(t,x) \;-\; ru(t,x) \;=\; 0,$$

and $u(T,x) = g(x)$.

Then $\widetilde{V}_t := e^{-rt}u(t,S_t)$ satisfies

$$\widetilde{V}_t = u(0,S_0) + \int_0^t \partial_x u(s,S_s)d\widetilde{S}_s.$$

Further, with the dynamic trading strategy $\phi_t = \partial_x u(t,S_t)$, and initial wealth $\Pi_0 = u(0,S_0)$, one has

$$\widetilde{\Pi}_t = \Pi_0 + \int_0^t \phi_s d\widetilde{S}_s = u(0,S_0) + \int_0^t \partial_x u(s,S_s)d\widetilde{S}_s.$$

It follows that

$$\widetilde{\Pi}_t = \widetilde{V}_t = e^{-rt}u(t,S_t) \quad \Leftrightarrow \quad \Pi_t = u(t,S_t).$$

Black-Scholes Formula: Delta hedging

Since $u(T,x) = g(x)$, one has

$$\Pi_T \;=\; u(T, S_T) \;=\; g(S_T),$$

i.e. the perfect replication of the payoff of the call option.

(1) Solve the Black-Scholes PDE

$$\partial_t u(t,x) \;+\; rx\partial_x u(t,x) \;+\; \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} u(t,x) \;-\; ru(t,x) \;=\; 0,$$

with terminal condition $u(T,x) = g(x)$.

(2) Construct a perfect replication portfolio Π, i.e. with initial wealth $\Pi_0 = u(0, S_0)$ and dynamic trading strategy $\phi_t = \partial_x u(t, S_t)$, one has

$$\Pi_T = g(S_T).$$

(3) The call option price is given by

$$\Pi_0 = u(0, S_0).$$

The Black-Scholes equation

Theorem 2.1

Assume that u satisfies the Black-Scholes equation

$$\partial_t u + \frac{1}{2}\sigma^2 x^2 \partial_{xx}^2 u + rx\partial_x u - ru = 0,$$

with terminal condition $u(T, x) = g(x)$.

Then, for option with payoff $g(S_T)$ at maturity time T,

its prices at time 0 is given by $u(0, S_0)$,

the corresponding replication strategy is $\phi_t = \partial_x u(t, S_t)$.

Girsanov Theorem

Recall that S satisfies the dynamic

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where B is a standard Brownian motion. Let us define

$$B_t^{\mathbb{Q}} := B_t + \lambda t, \text{ with } \lambda := \frac{\mu - r}{\sigma},$$

so that

$$dS_t = r S_t dt + \sigma S_t dB_t^{\mathbb{Q}}.$$

Theorem 2.2 (Girsanov Theorem (Not required for the exam))

Let us define a probability measure $\mathbb{Q} : \mathcal{F} \longrightarrow \mathbb{R}$ *by*

$$\mathbb{E}^{\mathbb{Q}}[\xi] := \mathbb{E}\Big[\xi \exp\big(-\lambda^2 T/2 - \lambda B_T\big)\Big], \quad \text{for all (bounded) r.v. } \xi.$$

Then, the process $B^{\mathbb{Q}}$ *is a standard Brownian motion under the probability measure* $\mathbb{Q}$.

Continuous-Time Risk-Neutral Valuation

In the risk neutral world (under the risk neutral probability $\mathbb{Q}$), the stock price follows:

$$\begin{aligned} \mathrm{d}S_t &= rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}B_t^{\mathbb{Q}}, \\ S_T &= S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma B_T^{\mathbb{Q}}}, \\ \ln(S_T) &\sim^{\mathbb{Q}} N\left(\ln(S_0) + (r - \tfrac{1}{2}\sigma^2)T,\ \sigma^2 T\right). \end{aligned}$$

Theorem 2.3

For option with payoff $g(S_T)$, its price at time 0 is given by

$$u(0, S_0) \;=\; \mathbb{E}^{\mathbb{Q}}\left[e^{-rT} g(S_T)\right].$$