MATH4210: Financial Mathematics

IV: Continuous Time Market, Part A: a martingale approach

Risk-free asset: the interest rate

• Discrete-time market: let $t_k := k\Delta t$, and the interest rate be $r \geq 0$, then an investment of $1\$$ at time $t_0 = 0$ leads to

$$S^0_{t_0} = 1, \quad S^0_{t_k} = (1 + r\Delta t)^k, \quad \text{for all } k \geq 1.$$

• Continuous-time market: let $\Delta t := t/k$, and $k \longrightarrow \infty$ so that $\Delta t \longrightarrow 0$, then

$$S^0_t \;=\; \lim_{k\to\infty} (1 + r\Delta t)^k \;=\; e^{rt}.$$

Risk-free asset: the interest rate

Recall that

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

n (Compounding frequency)	$(1+1/n)^n$ (value of \$1 in one year)
1	2
2	2.25
4	2.44141
12	2.61304
52	2.66373
365	2.69260
10000	2.71815
1000000	2.71828

Risky asset: the Black-Scholes model

The stock price $(S_t)_{0\le t\le T}$ follows the Black-Scholes model:

$$S_t \;=\; S_0 \exp\big((\mu-\sigma^2/2)t+\sigma B_t\big), \quad t\ge 0,$$

where B is a standard Brownian motion.

Remark 1

One has

$$S_t = S_0 e^{\mu t}\exp\Big(-\frac{\sigma^2}{2}t+\sigma B_t\Big),$$

so that

$$\mathbb{E}\big[S_t\big] = S_0 e^{\mu t}, \quad t\ge 0.$$

Remark 2

One can also show that the Black-Scholes model is the limit of the binomial model when $\Delta t \longrightarrow 0$.

Dynamic trading

Dynamic trading: let $t_k := k\Delta t$, risky asset price $(S_{t_k})_{k\geq 0}$, interest rate $r \geq 0$.

Discrete-time dynamic trading between t_k and t_{k+1}:

$$\begin{aligned}
\Pi_{t_{k+1}} &= \phi_{t_k} S_{t_{k+1}} + \big(\Pi_{t_k} - \phi_{t_k} S_{t_k}\big)(1 + r\Delta t) \\
&= \Pi_{t_k} + \big(\Pi_{t_k} - \phi_{t_k} S_{t_k}\big) r\Delta t + \phi_{t_k}\big(S_{t_{k+1}} - S_{t_k}\big).
\end{aligned}$$

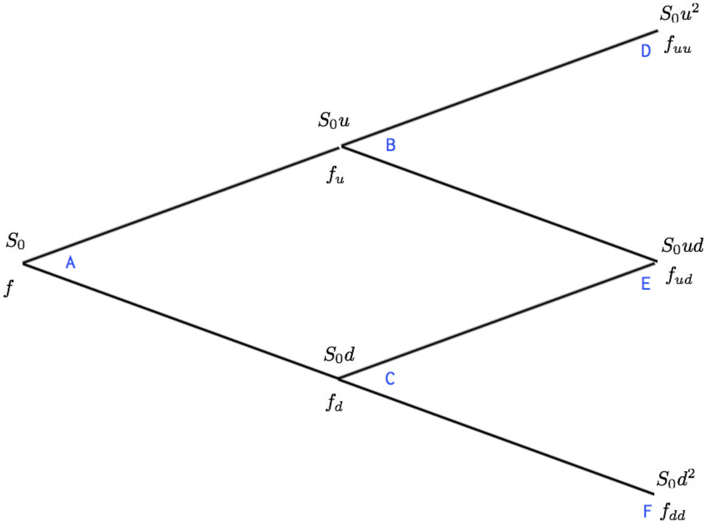
Then

$$\Pi_{t_n} = \Pi_0 + \sum_{k=0}^{n-1} \big(\Pi_{t_k} - \phi_{t_k} S_{t_k}\big) r\Delta t + \sum_{k=0}^{n-1} \phi_{t_k}\big(S_{t_{k+1}} - S_{t_k}\big).$$

The continuous-time limit:

$$\Pi_T = \Pi_0 + \int_0^T (\Pi_t - \phi_t S_t) r dt + \int_0^T \phi_t dS_t.$$

Pricing by the martingale approach: discrete time market



Pricing by the martingale approach: discrete time market

The risk-neutral probability

$$q = \frac{1 + r\Delta t - d}{u - d}.$$

Price of the derivative option:

$$\begin{aligned} f_u &= (1 + r\Delta t)^{-1}\big(q f_{uu} + (1-q) f_{ud}\big) = \mathbb{E}^{\mathbb{Q}}[(1+r\Delta t)^{-1} f_{t_2} | S_{t_1} = S_{t_1}^u], \\ f_d &= (1 + r\Delta t)^{-1}\big(q f_{ud} + (1-q) f_{dd}\big) = \mathbb{E}^{\mathbb{Q}}[(1+r\Delta t)^{-1} f_{t_2} | S_{t_1} = S_{t_1}^d], \\ f_{t_0} &= \mathbb{E}^{\mathbb{Q}}[(1+r\Delta t)^{-2} f_{t_2} | S_{t_0} = S_0], \end{aligned}$$

It follows that the following discounted process are martingales under $\mathbb{Q}$:

$$\big((1+r\Delta t)^{-k} S_{t_k}\big)_{k=0,1,2}, \qquad \big((1+r\Delta t)^{-k} f_{t_k}\big)_{k=0,1,2}.$$

The martingale approach: continuous time

- Pricing rule by the martingale approach: The risky asset follows the dynamic:

$$S_t = S_0 \exp\left((r - \sigma^2/2)t + \sigma B_t^{\mathbb{Q}}\right), \ t \geq 0,$$

where $B^{\mathbb{Q}}$ is a Brownian motion under the risk neutral probability $\mathbb{Q}$. For an option with payoff function $g(S_T)$, the option price is given by

$$u(t, S_t) := \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} g(S_T) \middle| S_t\right],$$

so that the following discounted process are martingales:

$$\left(e^{-rt} S_t\right)_{t \in [0,T]}, \quad \left(e^{-rt} u(t, S_t)\right)_{t \in [0,T]}.$$

Remark 3

We will justify this pricing rule later by replication argument.

Black-Scholes Formula for call, put options

More generally, one has:

Theorem 2.1

The the Black-Scholes formula for European call option is

$$C_E(t, S_t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2),$$

and the Black-Scholes formula for European put option is

$$P_E(t, S_t) = Ke^{-r(T-t)} N(-d_2) - S_t N(-d_1),$$

where

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

and

$$d_2 = \frac{\ln(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Black-Scholes model: the PDE

Theorem 2.2

Let $u(t,s)$ *denote the price of a vanilla European option with payoff* $g(S_T)$ *knowing that* $S_t = s$, *i.e.*

$$u(t,s) \ := \ \mathbb{E}^{\mathbb{Q}}\Big[g(S_T) e^{-r(T-t)} \Big| S_t = s \Big].$$

Then u *is the solution to the PDE (partial differential equation):*

$$\begin{cases} \partial_t u(t,s) + \frac{1}{2}\sigma^2 s^2 \partial^2_{ss} u(t,s) + rs\partial_s u(t,s) - ru(t,s) = 0, \\ u(T,s) = g(s). \end{cases}$$

- Remark: let $v(t,s) := u(t,s)e^{-rt}$, then v is solution to the PDE:

$$\partial_t v(t,s) + \frac{1}{2}\sigma^2 s^2 \partial^2_{ss} v(t,s) + rs\partial_s v(t,s) = 0.$$

Call option price properties

The Black-Scholes formula for the vanilla European call option has the following properties

- Delta: $\frac{\partial C_E}{\partial S} > 0$. (Note that $\Delta = \frac{\partial C_E}{\partial S}$.)
- Theta: $\frac{\partial C_E}{\partial (T-t)} > 0$.
- Rho: $\frac{\partial C_E}{\partial r} > 0$.
- Vega: $\frac{\partial C_E}{\partial \sigma} > 0$.
- Gamma: $\Gamma = \frac{\partial^2 C_E}{\partial S^2}$.
- $\frac{\partial C_E}{\partial K} < 0$.

Call option price properties

		Call	Put
Delta	$\frac{\partial V}{\partial S}$	$N(d_1)$	$-N(-d_1) = N(d_1) - 1$
Gamma	$\frac{\partial^2 V}{\partial S^2}$	$\frac{N'(d_1)}{S\sigma\sqrt{T-t}}$	
Vega	$\frac{\partial V}{\partial \sigma}$	$SN'(d_1)\sqrt{T-t}$	
Theta	$\frac{\partial V}{\partial t}$	$-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$	$-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_2)$
Rho	$\frac{\partial V}{\partial r}$	$K(T-t)e^{-r(T-t)}N(d_2)$	$-K(T-t)e^{-r(T-t)}N(-d_2)$

Call option price properties

Monotonicity in the factors:

increasing in	call option price	intuitive reason
$S(t)$	increases	potential payoff increases
K	decreases	potential payoff decreases
$T-t$	increases	more *"time value"*
r	increases	present value of fees K decreases
volatility σ	increases	risk increases

Greek Letters

Because the price C_E satisfies

$$\frac{\partial C_E}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_E}{\partial S^2} + rS\frac{\partial C_E}{\partial S} - rC_E = 0,$$

we derive that

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta = rC_E.$$