MATH4210: Financial Mathematics

III. Discrete Time Market

Discrete time model

- $0 = t_0 < t_1 < \cdots < t_n = T$, with $t_k := k\Delta t = k\frac{T}{n}$.
- One risky asset whose price is given by $S = (S_{t_k})_{0 \leq k \leq n}$.
- One risk-free asset with "interest rate" $r \geq 0$:

  investment of $1\$$ at time t_k gives $(1 + r\Delta t)\$$ at time t_{k+1}.

Remark: for each k, S_{t_k} is a random variable, whose distribution will be precised later.

Dynamic trading strategy

- Let Π_{t_0} represent the initial wealth of the portfolio at time t_0.
- Let ϕ_{t_0} be the number of risky asset in portfolio from period t_0 to t_1.
- Then $\Pi_{t_0} - \phi_{t_0} S_{t_0}$ is the amount of cash (deposit in bank account).
- Let Π_{t_1} be the total wealth of the portfolio at time t_1, then

$$\begin{aligned}\Pi_{t_1} &= \phi_{t_0} S_{t_1} + \big(\Pi_{t_0} - \phi_{t_0} S_{t_0}\big)(1 + r\Delta t) \\ &= \Pi_{t_0}(1 + r\Delta t) + \phi_{t_0}\big(S_{t_1} - S_{t_0}(1 + r\Delta t)\big).\end{aligned}$$

Dynamic trading strategy

- Let Π_{t_k} represent the wealth of the portfolio at time t_k.
- Let ϕ_{t_k} be the number of risky asset in portfolio for period t_k to t_{k+1}.
- Then $\Pi_{t_k} - \phi_{t_k} S_{t_k}$ is the amount of cash (deposit in bank account) for period t_k to t_{k+1}.
- Let $\Pi_{t_{k+1}}$ be the total wealth of the portfolio at time t_{k+1}, then

$$\begin{aligned}
\Pi_{t_{k+1}} &= \phi_{t_k} S_{t_{k+1}} + \left(\Pi_{t_k} - \phi_{t_k} S_{t_k}\right)(1 + r\Delta t) \\
&= \Pi_{t_k}(1 + r\Delta t) + \phi_{t_k}\left(S_{t_{k+1}} - S_{t_k}(1 + r\Delta t)\right).
\end{aligned}$$

Dynamic trading strategy

Let us consider the discounted value (present value) of S and Π, i.e.

$$\widetilde{S}_{t_k} := S_{t_k}(1 + r\Delta t)^{-k}, \quad \widetilde{\Pi}_{t_k} := \Pi_{t_k}(1 + r\Delta t)^{-k},$$

it follows that

$$\widetilde{\Pi}_{t_1} = \widetilde{\Pi}_{t_0} + \phi_{t_0}\left(\widetilde{S}_{t_1} - \widetilde{S}_{t_0}\right),$$

and more generally,

$$\widetilde{\Pi}_{t_{k+1}} = \widetilde{\Pi}_{t_k} + \phi_{t_k}\left(\widetilde{S}_{t_{k+1}} - \widetilde{S}_{t_k}\right).$$

Dynamic trading strategy

We then obtain

$$\widetilde{\Pi}_{t_k} \;=\; \widetilde{\Pi}_{t_0} + \sum_{i=0}^{k-1} \phi_{t_i} \big(\widetilde{S}_{t_{i+1}} - \widetilde{S}_{t_i}\big) \;=\; \Pi_{t_0} + \sum_{i=0}^{k-1} \phi_{t_i} \big(\widetilde{S}_{t_{i+1}} - \widetilde{S}_{t_i}\big),$$

and hence

$$\Pi_{t_k} \;=\; (1 + r\Delta t)^k \Big(\Pi_{t_0} + \sum_{i=0}^{k-1} \phi_{t_i} \big(\widetilde{S}_{t_{i+1}} - \widetilde{S}_{t_i}\big)\Big).$$

Conclusion: Value of the portfolio is completely determined by its initial value Π_{t_0} and the dynamic trading strategy ϕ. Let us denote

$$\Pi_{t_k}^{x,\phi} \;:=\; (1 + r\Delta t)^k \Big(x + \sum_{i=0}^{k-1} \phi_{t_i} \big(\widetilde{S}_{t_{i+1}} - \widetilde{S}_{t_i}\big)\Big).$$

Option Pricing by replication

Proposition 1.1

Let $G(S_{t_0}, S_{t_1}, \cdots, S_{t_n})$ be the payoff of a derivative option at maturity t_n. Assume that there is some (x, ϕ) such that

$$G(S_{t_0}, S_{t_1}, \cdots, S_{t_n}) \;=\; \Pi_{t_n}^{x,\phi}, \text{ a.s.}$$

then the price of the option should be

$$x.$$

Binomial Trees

Use of the model

In finance, the binomial options pricing model provides a generalizable numerical method for the valuation of options. The binomial model was first proposed by Cox, Ross and Rubinstein (1979). Essentially, the model uses a "discrete-time" (lattice based) model of the varying price over time of the underlying financial instrument.

For options with several sources of uncertainty (e.g., real options) and for options with complicated features (e.g., Asian options), binomial methods are less practical due to several difficulties, and Monte Carlo option models are commonly used instead. Monte Carlo simulation is computationally time-consuming, however.

Binomial Trees

Methodology

The binomial pricing model traces the evolution of the option's key underlying variables in discrete-time. This is done by means of a binomial lattice (tree), for a number of time steps between the valuation and expiration dates. Each node in the lattice, represents a possible price of the underlying at a given point in time.

Valuation is performed iteratively, starting at each of the nodes at maturity date, and then working backwards through the tree towards the first node (valuation date). The value computed at each stage is the value of the option at that point in time.

Option valuation using this method is a three-step process:

1. price tree generation
2. calculation of option value at each final node
3. sequential calculation of the option value at each preceding node

One-Step Binomial Model

A European call option with strike $K = \$21$ at time $T = 1$. Interest rate $r = 0$. The current stock price is $S_0 = \$20$. Suppose that we know S_1 will be either $\$22$ or $\$18$.

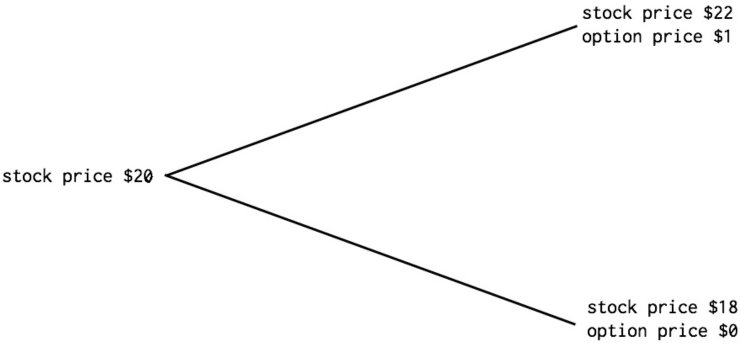
- Call option with payoff $(S_T - K)_+$, $T = 1$, $K = 21$.
- Interest rate $r = 0$.
- $S_0 = 20$ and for $p = 50\%$,

$$\mathbb{P}[S_1 = 22] = p, \qquad \mathbb{P}[S_1 = 18] = 1 - p.$$

Remark: At time T, the option price (payoff) is $\$1$ if $S_T = \$22$; and the option price (payoff) is $\$0$ if $S_T = \$18$.

One-Step Binomial Model

A European call option with strike $K = \$21$ at time $T = 1$. Continuously compounded interest rate $r = 0$. The current stock price is $S_0 = \$20$. Suppose we know S_1 will be either $\$22$ or $\$18$.



One-Step Binomial Model

Value of the portfolio of the dynamic trading

$$\Pi_1^{x,\phi} = x(1+r) + \phi_0\big(S_1 - S_0(1+r)\big).$$

At time $T = 1$, two possibilities:

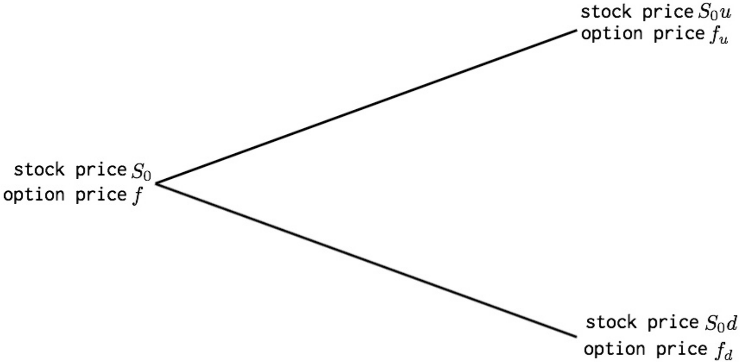
- if $S_1 = \$22$, option payoff is $\$1$,
  value of the portfolio $\Pi_1^{x,\phi} = x + 2\phi_0$.
- if $S_1 = \$18$, option payoff is $\$0$,
  value of the portfolio $\Pi_1^{x,\phi} = x - 2\phi_0$.

Replication leads to

$$x = 0.5 \quad \text{and} \quad \phi_0 = 0.25.$$

One-Step Binomial Model

The current stock price is S_0 and price of an option on it is f. Suppose the stock price at maturity of the option can either move up to $S_0 u$ where $u > 1$ or down to $S_0 d$ where $d < 1$. Suppose the payoffs of the option are f_u and f_d when the stock price are $S_0 u$ and $S_0 d$, respectively.



One-Step Binomial Model

As before, we consider a portfolio (with initial wealth x):

- Long ϕ_0 share of stock,
- Long $x - \phi_0 S_0$ cash.

If there is an up movement in the stock price, then the value of the portfolio at maturity is

$$\Pi_1^{x,\phi} = (x - \phi_0 S_0)(1 + r\Delta t) + \phi_0 S_0 u.$$

If there is a down movement in the stock price, then the value becomes

$$\Pi_1^{x,\phi} = (x - \phi_0 S_0)(1 + r\Delta t) + \phi_0 S_0 d.$$

To replication the payoff of the option f_u and f_d respectively, one should have

$$0 = (x - \phi_0 S_0)(1 + r\Delta t) + \phi_0 S_0 u - f_u = (x - \phi_0 S_0)(1 + r\Delta t) + \phi_0 S_0 d - f_d,$$

which leads to

$$\phi_0 = \frac{f_u - f_d}{S_0 u - S_0 d} \quad \text{and} \quad x = (1 + r\Delta t)^{-1}\Big(f_u - \phi_0 S_0\big(u - (1 + r\Delta t)\big)\Big).$$

One-Step Binomial Model

By replication argument, it follows that

$$V_{t_0} := x = (1+r\Delta t)^{-1}\Big(f_u - \phi_0 S_0\big(u-(1+r\Delta t)\big)\Big) = (1+r\Delta t)^{-1}\big(qf_u+(1-q)f_d\big)$$

where

$$q = \frac{(1+r\Delta t)-d}{u-d}.$$

Assume that $q \in (0,1)$, and let $\mathbb{Q}$ be such that

$$\begin{cases} \mathbb{Q}[S_{t_1} = uS_0] & = & \mathbb{Q}[V_{t_1} = f_u] = q, \\ \mathbb{Q}[S_{t_1} = dS_0] & = & \mathbb{Q}[V_{t_1} = f_d] = 1-q, \end{cases}$$

then

$$V_{t_0} = \mathbb{E}^{\mathbb{Q}}\big[(1+r\Delta t)^{-1}V_{t_1}\big].$$

One-Step Binomial Model

In the previous numerical example, $u = 22/20 = 1.1$, $d = 18/20 = 0.9$, $r = 0$, $T = 1$, $f_u = 1$ and $f_d = 0$. So we have

$$q = \frac{1 + r\Delta t - 0.9}{1.1 - 0.9} = 0.5$$

and the option price is

$$V_{t_0} = (1 + r\Delta t)(0.5 * 1 + 0.5 * 0) = 0.5,$$

and the trading strategy is

$$\phi_0 = \frac{f_u - f_d}{S_0 u - S_0 d} = 0.25.$$

Risk Neutral Valuation

The condition

$$q \in (0,1) \quad \Leftrightarrow \quad d < (1 + r\Delta t) < u.$$

Proposition 1.2

If $(1 + r\Delta t) \le d < u$, or $d < u \le (1 + r\Delta t)$, then there is an arbitrage opportunity (i.e. a strategy (or portfolio) such that $\Pi_{t_0} = 0$, $\mathbb{P}[\Pi_{t_1} \ge 0] = 1$ and $\mathbb{P}[\Pi_{t_1} > 0] > 0$).

Conclusion: under the no-arbitrage condition, one has $q \in (0,1)$.

Risk Neutral Valuation

Option pricing formula

$$V_{t_0} = (1 + r\Delta t)(qf_u + (1-q)f_d), \quad \text{with} \;\; q = \frac{(1+r\Delta t)-d}{u-d},$$

does not involve any assumptions about the probabilities of up (and down) movements in the stock price:

$$p = \mathbb{P}[S_{t_1} = uS_0].$$

However, it is natural to interpret q as the probability of up movement. Then the option price is present value of the expected payoff of the option in the world. We call it *risk neutral world* if we set the probability of an up movement in the stock price to q, i.e.

$$\mathbb{Q}[S_{t_1} = uS_0] = q.$$

Risk Neutral Valuation

In the risk neutral world, the expected stock price at time $T = t_1$ is given by

$$\mathbb{E}^{\mathbb{Q}}[S_T] = qS_0u + (1-q)S_0d = S_0(1 + r\Delta t).$$

In the risk neutral world, the expected option price at time $T = t_1$ is given by

$$\mathbb{E}^{\mathbb{Q}}[V_T] = qf_u + (1-q)f_d = f(1 + r\Delta t).$$

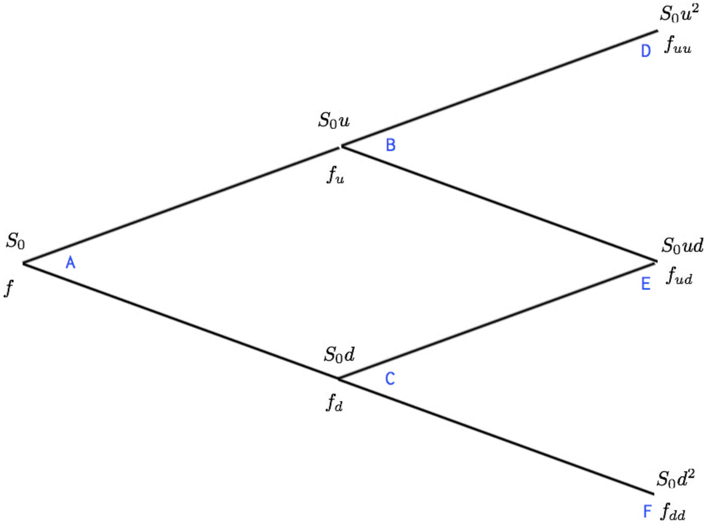
In the risk neutral world, the expected return on all securities is the risk free interest rate. Setting the probability of an up movement in the stock price to q is therefore equivalent to assuming that the return on the stock equals risk free rate.

Risk Neutral Valuation

Risk neutral valuation principle

We can assume the world is risk neutral when pricing options. The resulting prices are correct not just in a risk neutral world but in other worlds as well.

Two-Steps Binomial Model



Two-Step Binomial Model

Now let us consider node B. Note that B,D,E is a one step binomial model.
The length of time step is Δt, we have

$$V_{t_1}^{u} = f_u = (1 + r\Delta t)^{-1}(qf_{uu} + (1 - q)f_{ud}),$$

where

$$q = \frac{(1 + r\Delta t) - d}{u - d}.$$

The dynamic trading strategy is given by

$$\phi_{t_1}^{u} \ := \ \frac{f_{uu} - f_{ud}}{S_0 u^2 - S_0 ud}$$

Remark: $\phi = (\phi_{t_0}, \phi_{t_1})$ represents the number of stocks in the portfolio, the total value of the portfolio is determined by ϕ.

Two-Steps Binomial Model

Repeated the idea,

$$
\begin{aligned}
V_{t_1}^u := f_u &= (1+r\Delta t)^{-1}(qf_{uu} + (1-q)f_{ud}), \\
V_{t_1}^d := f_d &= (1+r\Delta t)^{-1}(qf_{ud} + (1-q)f_{dd}), \\
\phi_{t_1}^d &= \frac{f_{ud} - f_{dd}}{S_0 ud - S_0 dd}, \\
V_{t_0} := f &= (1+r\Delta t)^{-1}(qf_u + (1-q)f_d), \\
\phi_{t_0} &= \frac{f_u - f_d}{S_0 u - S_0 d}.
\end{aligned}
$$

Finally, we get

$$
V_{t_0} = (1+r\Delta t)^{-2}(q^2 f_{uu} + 2q(1-q)f_{ud} + (1-q)^2 f_{dd}).
$$

Two-Steps Binomial Model

Example 1.1

Let us consider consider a 2-year European put with strike price $\$52$ on a stock whose current price is $\$50$. We suppose there are two time steps of 1 year, and in each time step, the stock price either moves up by 20% or moves down by 20%. We also suppose the risk-free interest rate is 5%.

Two-Steps Binomial Model

In this case $u = 1.2$, $d = 0.8$, $\Delta t = 1$, and $r = 5\%$. Then q is given by

$$q = \frac{(1 + r\Delta t) - d}{u - d} = \frac{1 + 0.05 - 0.8}{1.2 - 0.8} = 5/8.$$

The possible final stock prices are $50 * 1.2^2 = 72$, $50 * 1.2 * 0.8 = 48$, $50 * 0.8^2 = 32$. Therefore, $f_{uu} = 0$, $f_{ud} = 4$, and $f_{dd} = 20$, and one can then compute the price as well as the replication strategy.

Two-Steps Binomial Model

In summary, one has

$$\begin{aligned} V_{t_1}^u &= \mathbb{E}^{\mathbb{Q}}\left[V_{t_2}(1+r\Delta t)^{-1} \middle| S_{t_1} = uS_0\right], \\ V_{t_1}^d &= \mathbb{E}^{\mathbb{Q}}\left[V_{t_2}(1+r\Delta t)^{-1} \middle| S_{t_1} = dS_0\right], \\ V_{t_0} &= \mathbb{E}^{\mathbb{Q}}\left[V_{t_1}(1+r\Delta t)^{-1}\right]. \end{aligned}$$

In other words, the following discounted price processes are martingales:

$$\left((1+r\Delta t)^{-k} S_{t_k}\right)_{k=0,1,2}, \quad \left((1+r\Delta t)^{-k} V_{t_k}\right)_{k=0,1,2}.$$

American Option:One-Step Binomial Model

- Let us consider an American option, whose payoff is $C(S_{t_k})$, $k = 0, 1$.
  - If the option is exercised at time t_0, the option holder receives $C(S_{t_0})$,
  - otherwise, the option holder wait until time t_1 to receive $C(S_{t_1})$.

- The value of the option at time t_0 will be given by

$$\max\Big(C(S_{t_0}),\ \mathbb{E}^{\mathbb{Q}}[C(S_{t_1})(1+r\Delta t)^{-1}]\Big).$$

American Option:Two-Steps Binomial Model

Suppose the payoff function of the American option is $C(S_{t_k})$ if the option is exercised at time t_k (stock price is S_{t_k}), $k = 0, 1, 2$. Now let us consider node B.

- If the option holder does not exercise the option, then it is same as a European option, so the value (at node B) is

$$(1 + r\Delta t)^{-1}(qf_{uu} + (1 - q)f_{ud}).$$

- If the holder exercises it, the payoff is $C(S_0u)$.

American Option: Two-Steps Binomial Model

So at node B, the value of the option is the maximum of the two choices,

$$f_u = \max\left(C(S_0u),\ (1+r\Delta t)^{-1}(qf_{uu}+(1-q)f_{ud})\right).$$

The replication trading strategy is

$$\phi_{t_1}^u = \begin{cases} 0 & \text{if it is exercised at node B,} \\ \frac{f_{uu}-f_{ud}}{S_0u^2-S_0ud} & \text{if it is not exercised at node B.} \end{cases}$$

American Option: Two-Steps Binomial Model

Repeated the idea,

$$\begin{aligned} f_u &= \max\{C(S_0u), (1+r\Delta t)^{-1}(qf_{uu}+(1-q)f_{ud})\}, \\ f_d &= \max\{C(S_0d), (1+r\Delta t)^{-1}(qf_{ud}+(1-q)f_{dd})\}, \\ f &= \max\{C(S_0), (1+r\Delta t)^{-1}(qf_u+(1-q)f_d)\}. \end{aligned}$$

$$\phi_{t_1}^u = \begin{cases} 0 & \text{if it is exercised at node B,} \\ \frac{f_{uu}-f_{ud}}{S_0u^2-S_0ud} & \text{if it is not exercised at node B.} \end{cases}$$

$$\phi_{t_1}^d = \begin{cases} 0 & \text{if it is exercised at node C,} \\ \frac{f_{ud}-f_{dd}}{S_0ud-S_0dd} & \text{if it is not exercised at node C.} \end{cases}$$

$$\phi_{t_0} = \begin{cases} 0 & \text{if it is exercised at node A,} \\ \frac{f_u-f_d}{S_0u-S_0d} & \text{if it is not exercised at node A.} \end{cases}$$

American Option: Two-Steps Binomial Model

Example 1.2

Let us consider consider a 2-year American put with strike price $\$52$ *on a stock whose current price is* $\$50$. *We suppose there are two time steps of 1 year, and in each time step, the stock price either moves up by* 20% *or moves down by* 20%. *We also suppose the risk-free interest rate is* 5%.

American Option: Two-Steps Binomial Model

In this case $u = 1.2$, $d = 0.8$, $\Delta t = 1$, and $r = 5\%$. Then q is given by

$$q = \frac{(1 + r\Delta t) - d}{u - d} = \frac{1 + 0.05 - 0.8}{1.2 - 0.8} = 0.625.$$

So

$$\begin{aligned}
f_u &= \max\{0, 1.05^{-1}(0.625 * 0 + (1 - 0.625) * 4)\} = 1.429, \\
f_d &= \max\{12, 1.05^{-1}(0.625 * 4 + (1 - 0.625) * 20\} = 12, \\
f &= \max\{2, 1.05^{-1}(0.625 * 1.429 + (1 - 0.625) * 12)\} = 5.136.
\end{aligned}$$

It is better to exercise the option at node C than to hold it.

Hedging of Claims

- **Example:** Consider a two-step binomial tree model for an European call option with $S_0 = \$100$, $u = 1.1$, $d = 0.9$ and one-step interest $r = 0.05$. Find the price and the replicating strategy of this European call option with strike price $K = \$95$.

- **Example:** Consider a three-step binomial tree model with $S_0 = \$100$, $u = 1.1$, $d = 0.9$, $r = 0$. We consider an American put option with strike price $K = \$100$ and maturity $T = t_3$. Compute the hedging strategy along the *ddd* scenrio.

More Examples

- **Example 1:** In a two step binomial tree model with one step interest $r = 0.05$, $S_0 = 100$, $u = 1.1$, $d = 0.9$, consider an contingent claim that expires after two years and payoff is the value of the squared stock price $(S(T))^2$, if the stock price $S(T)$ is strictly higher than 100 when the option is exercised; otherwise, the option pays 0.
  (1) Find the initial price and the replication strategy of the European version of the above option.
  (2) In the same market model above, find the initial price and the replication strategy of the American version of the above option.

- **Example 2:** A **Lookback call** is identical to a standard European call, except that the strike price is not set in advance, but is equal to the minimum price experienced by the underlying asset during the life of the call. Suppose the stock price $S_0 = 100$, $u = 1.1$, $d = 0.9$ in each of the next two years, and one step interest $r = 0.05$. What is the price and the replication strategy of a two-year Lookback call option ?

More Examples

- **Example 3:** Suppose you are given a two step binomial tree model with the following: $S_0 = 100$, $u = 1.04$, $d = 0.96$, $r = 0.05$. Consider a two period **Asian call option** where the averaging is done over all three prices observed, i.e., the initial price, the price after one period, and the price after two periods.
  (1) Suppose the option is an average-price Asian option with a strike of 100. Find the initial price and the replication strategy.
  (2) Suppose the option is an average-strike Asian option. Find the initial price and the replication strategy.

- **Example 4.** Consider a two step binomial tree with the following parameters: $S_0 = 100$, $u = 1.1$, $d = 0.9$ and $r = 0.05$. Find the prices and the replication strategy of
  (1) A European **knock-out call** option with a strike price of 95 and a barrier of 90.
  (2) A European **knock-in call** option with a strike price of 95 and a barrier of 90.
  (3) A European call option with a strike price 95.