Math 4210 Assignment 2 Solution

February 28, 2024

Problem 1. (30pts)

a) (10pts) Notice that

$$\begin{aligned}\mathbb{E}\big[e^{-r(T-t)}f(B_T)|B_t=x\big] =&\mathbb{E}\big[e^{-r(T-t)}f(B_T-B_t+x)\big]\\ =&\mathbb{E}\big[e^{-r(T-t)}f(Y+x)\big] \qquad \text{(2pts)}\end{aligned}$$

where $Y\sim N(0,T-t)$ with probability density function

$$p_Y(y)=\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{y^2}{2(T-t)}}.$$

Then

$$\begin{aligned}u(t,x)=&\int_{\mathbb{R}}e^{-r(T-t)}(y+x-K)_+p_Y(y)dy\\ =&e^{-r(T-t)}\int_{K-x}^{\infty}(y+x-K)p_Y(y)dy\\ =&e^{-r(T-t)}\Big(\int_{K-x}^{\infty}yp_Y(y)dy+(x-K)\int_{K-x}^{\infty}p_Y(y)dy\Big)\\ =&e^{-r(T-t)}\big(I_1+(x-K)I_2\big)\end{aligned}$$

Notice that $p_Y'(y)=-\frac{yp_Y(y)}{T-t}$, we have

$$I_1=-(T-t)\int_{K-x}^{\infty}-\frac{y}{T-t}p_Y(y)dy=(T-t)p_Y(K-x). \qquad \text{(3pts)}$$

On the other hand,

$$I_2=\mathbb{P}[Y\geq K-x]=\mathbb{P}[Z\geq\frac{K-x}{\sqrt{T-t}}]=\Phi(\frac{x-K}{\sqrt{T-t}}) \qquad \text{(3pts)}$$

where $Z\sim N(0,1)$ and Φ is the cumulative distribution function of Z. So we conclude

$$u(t,x)=e^{-r(T-t)}\Big(\frac{\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{(K-x)^2}{2(T-t)}}+(x-K)\Phi(\frac{x-K}{\sqrt{T-t}})\Big)$$

or equivalently

$$u(t,x) = e^{-r(T-t)}\Big(\sqrt{T-t}p_Z(\frac{x-K}{\sqrt{T-t}}) + (x-K)\Phi(\frac{x-K}{\sqrt{T-t}})\Big) \qquad (2\text{pts})$$

where $p_Z(z) := \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$ is the probability density function of Z.

b) (15pts) By chain rule and product rule:

$$\begin{aligned}\partial_t u(t,x) =& ru(t,x) + e^{-r(T-t)}\Big(- \frac{1}{2\sqrt{T-t}}p_Z(\frac{x-K}{\sqrt{T-t}}) \\ &+ \sqrt{T-t}(-\frac{x-K}{\sqrt{T-t}})(x-K)(\frac{1}{2}(T-t)^{-3/2})p_Z(\frac{x-K}{\sqrt{T-t}})) \\ &+ (x-K)(x-K)(\frac{1}{2}(T-t)^{-3/2})p_Z(\frac{x-K}{\sqrt{T-t}})\Big) \\ =& ru(t,x) - e^{-r(T-t)}\frac{1}{2\sqrt{T-t}}p_Z(\frac{x-K}{\sqrt{T-t}}). \qquad (5\text{pts})\end{aligned}$$

$$\begin{aligned}\partial_x u(t,x) =& e^{-r(T-t)}\Big(\sqrt{T-t}\frac{1}{\sqrt{T-t}}(-\frac{x-K}{\sqrt{T-t}})p_Z(\frac{x-K}{\sqrt{T-t}}) + \Phi(\frac{x-K}{\sqrt{T-t}}) \\ &+ (x-K)\frac{1}{\sqrt{T-t}}p_Z(\frac{x-K}{\sqrt{T-t}})\Big) \\ =& e^{-r(T-t)}\Phi(\frac{x-K}{\sqrt{T-t}}). \qquad (5\text{pts})\end{aligned}$$

$$\partial^2_{xx}u(t,x) = e^{-r(T-t)}\frac{1}{\sqrt{T-t}}p_Z(\frac{x-K}{\sqrt{T-t}}). \qquad (5\text{pts})$$

c) (5pts) From b):

$$\partial_t u(t,x) + \frac{1}{2}\partial^2_{xx}u(t,x) - ru(t,x) = 0.$$

Problem 2. (30pts)

a) (5pts)

$$S_t = S_0 \exp\Big((r - \frac{\sigma^2}{2})t + \sigma B_t^{\mathbb{Q}}\Big), \text{ for } t \in [0,T].$$

b) (5pts)

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\big[e^{-rT}\mathbf{1}_{\{S_T\le K\}}\big] =& e^{-rT}\mathbb{Q}\big[S_T\le K\big]\\ =& e^{-rT}\mathbb{Q}\big[S_0\exp\Big((r-\frac{\sigma^2}{2})T+\sigma B_T^{\mathbb{Q}}\Big)\le K\big]\\ =& e^{-rT}\mathbb{Q}[\sigma B_T^{\mathbb{Q}}\le \log(K/S_0)-(r-\frac{\sigma^2}{2}T)]\\ =& e^{-rT}\mathbb{Q}[Z\le \frac{\log(K/S_0)-(r-\frac{\sigma^2}{2}T)}{\sigma\sqrt{T}}]\\ =& e^{-rT}\Phi(\frac{\log(K/S_0)-(r-\frac{\sigma^2}{2}T)}{\sigma\sqrt{T}})\end{aligned}$$

where $Z\sim N(0,1)$ under $\mathbb{Q}$.

c) (20pts) **Method 1.**

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}\big[e^{-rT}|S_T-K|\big]\\ =&\mathbb{E}^{\mathbb{Q}}\big[e^{-rT}(S_T-K)\mathbf{1}_{\{S_T\ge K\}}\big]+\mathbb{E}^{\mathbb{Q}}\big[e^{-rT}(K-S_T)\mathbf{1}_{\{S_T\le K\}}\big]\\ =&e^{-rT}\Big(E^{\mathbb{Q}}\big[S_T\mathbf{1}_{\{S_T\ge K\}}\big]-E^{\mathbb{Q}}\big[S_T\mathbf{1}_{\{S_T\le K\}}\big]-K\mathbb{Q}[S_T\ge K]+K\mathbb{Q}[S_T\le K]\Big)\end{aligned}$$

From b):

$$\begin{aligned}K\mathbb{Q}[S_T\le K]-K\mathbb{Q}[S_T\ge K] =&K\mathbb{Q}[S_T\le K]-K(1-\mathbb{Q}[S_T\le K])\\ =&2K\mathbb{Q}[S_T\le K]-K\\ =&2K\Phi(\frac{\log(K/S_0)-(r-\frac{\sigma^2}{2}T)}{\sigma\sqrt{T}})-K.\end{aligned}$$

(5pts)

Denote by $d=\frac{\log(K/S_0)-(r-\frac{\sigma^2}{2}T)}{\sigma\sqrt{T}}$ and $Y\sim N(0,T)$ under $\mathbb{Q}$ with probability density function p_Y given by $p_Y(y) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}}$. Then

$$
\begin{aligned}
E^{\mathbb{Q}}\big[S_T \mathbf{1}_{\{S_T \le K\}}\big] &= \int_{\mathbb{R}} S_0 \exp\Big((r - \frac{\sigma^2}{2})T + \sigma y\Big) \mathbf{1}_{y \le \sqrt{T}d} p_Y(y) dy \\
&= \int_{\mathbb{R}} S_0 \exp\Big((r - \frac{\sigma^2}{2})T + \sigma y\Big) \mathbf{1}_{y \le \sqrt{T}d} p_Y(y) dy \\
&= S_0 e^{(r - \frac{\sigma^2}{2})T} \int_{-\infty}^{\sqrt{T}d} \frac{1}{\sqrt{2\pi T}} e^{-(\frac{y^2}{2T} - \sigma y)} dy \\
&= S_0 e^{(r - \frac{\sigma^2}{2})T} \int_{-\infty}^{\sqrt{T}d} \frac{1}{\sqrt{2\pi T}} e^{-(\frac{(y-\sigma T)^2 - \sigma^2 T^2}{2T})} dy \\
&= S_0 e^{(r - \frac{\sigma^2}{2})T + \frac{\sigma^2 T}{2}} \int_{-\infty}^{\sqrt{T}d} \frac{1}{\sqrt{2\pi T}} e^{-(\frac{(y-\sigma T)^2}{2T})} dy \\
&= S_0 e^{rT} \mathbb{Q}[Y + \sigma T \le \sqrt{T}d] \\
&= S_0 e^{rT} \mathbb{Q}[Y \le \sqrt{T}d - \sigma T] \\
&= S_0 e^{rT} \mathbb{Q}[Z \le \frac{\sqrt{T}d - \sigma T}{\sqrt{T}}] \\
&= S_0 e^{rT} \Phi(d - \sigma\sqrt{T}) \qquad \text{(5pts)}
\end{aligned}
$$

Similarly, one computes

$$
E^{\mathbb{Q}}\big[S_T \mathbf{1}_{\{S_T \ge K\}}\big] = S_0 e^{rT} \Phi\Big(\frac{\log(S_0/K) + (r + \frac{\sigma^2}{2}T)}{\sigma\sqrt{T}}\Big) = S_0 e^{rT} \Phi(\sigma\sqrt{T} - d)
$$

(5pts)

To sum up:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\big[e^{-rT}|S_T - K|\big] &= S_0 \Phi(\sigma\sqrt{T} - d) - S_0 \Phi(d - \sigma\sqrt{T}) + 2Ke^{-rT}\Phi(d) - Ke^{-rT} \\
&= 2S_0 \Phi(\sigma\sqrt{T} - d) - S_0 + 2Ke^{-rT}\Phi(d) - Ke^{-rT}
\end{aligned}
$$

(5pts)

Method 2. It can be viewed as sum of a Vanilla call option and a Vanilla put option. By the formulae of option prices (5pts for each formula):

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\big[e^{-rT}|S_T - K|\big] &= S_0 \Phi(\sigma\sqrt{T} - d) - Ke^{-rT}\Phi(-d) + Ke^{-rT}\Phi(d) - S_0 \Phi(d - \sigma\sqrt{T}) \\
&= 2S_0 \Phi(\sigma\sqrt{T} - d) - S_0 + 2Ke^{-rT}\Phi(d) - Ke^{-rT}
\end{aligned}
$$

(10pts)