HW 4 A (Start Questions: 2 & 3)

1. Show the following "quasi-regularity" properties for outer-measure m^*: Let $m^*(A) < +\infty$. Then

   (i) $m^*(A) = \inf\{m(G) : \text{open } G \supseteq A\}$

   (ii) $\exists$ a G_δ-set $H := \bigcap_{n\in\mathbb{N}} G_n \supseteq A$ s.t. $m(H) = m^*(A)$ (where each G_n is open).

2*. Let $\{E_n : n \in \mathbb{N}\}$ be a sequence of measurable sets and let $E = \liminf E_n \left(:= \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} E_k = \bigcup_{n=1}^{\infty} T_n \right.$ where $\left. T_n := \bigcap_{k\geq n} E_k \ \forall n \right)$. Show that

$$m(E) \leq \liminf_n m(E_n)$$

via the following consideration

$$m(E) \leq \lim_n m(T_n) = \liminf_n m(T_n) \leq \liminf_n m(E_n).$$

(why I use $\leq$ in the first one rather than $=$)

3.* Let $I := [a,b]$ be a nonempty finite-length interval. Show that it intersects its "shifted interval $z + I$ and that $I \cup (z+I)$ is an interval with length $\leq (1+\delta)\cdot \ell(I)$ provided that $|z| \leq \delta \cdot \ell(I)$ and $\delta \in (0,1)$.