Star-Questions: 1, 5, 10, 11.

MATH4050 Real Analysis
Assignment 2

2024

There are ~~10~~ 11 questions in this assignment. The page number and question number for each question correspond to that in Royden's Real Analysis, 3rd or 4th edition.

In this assignment, $\{x_n\}$ and $\{y_n\}$ are sequences of real numbers. E is a subset of $\mathbb{R}$. $x \in \mathbb{R}$ is called a point of closure of E if each neighbourhood of x intersects E.

1.* (3rd: P.39, Q12)
Show that $x = \lim x_n$ if and only if every subsequence of $\{x_n\}$ has in turn a subsequence that converges to x.
Q: How about $x \in \{-\infty, \infty\}$?

2. (3rd: P.39, Q13)
Show that the real number l is the limit superior of the sequence $\{x_n\}$ if and only if (i) given $\varepsilon > 0$, $\exists n$ such that $x_k < l + \varepsilon$ for all $k \geq n$, and (ii) given $\varepsilon > 0$ and n, $\exists k \geq n$ such that $x_k > l - \varepsilon$.

3. (3rd: P.39, Q14)
Show that $\limsup x_n = \infty$ if and only if given Δ and n, $\exists k \geq n$ such that $x_k > \Delta$.
↓ definition is similarly defined as for bounded seq.

4. (3rd: P.39, Q15)
Show that $\liminf x_n \leq \limsup x_n$ and $\liminf x_n = \limsup x_n = l$ if and only if $l = \lim x_n$.

5.* (3rd: P.39, Q16)
Prove that
$$\limsup x_n + \liminf y_n \leq \limsup (x_n + y_n) \leq \limsup x_n + \limsup y_n.$$
provided the right and left sides are not of the form $\infty - \infty$.

6. (3rd: P.39, Q17)
Prove that if $x_n \geq 0$ and $y_n \geq 0$ $\forall n \in \mathbb{N}$, then
$$\limsup (x_n y_n) \leq (\limsup x_n)(\limsup y_n)$$
provided the product on the right is not of the form $0 \cdot \infty$.

7. (3rd: P.46, Q27)
Show that x is a point of closure of E if and only if there is a sequence $\{y_n\}$ with $y_n \in E$ and $x = \lim y_n$.

8. (3rd: P.46, Q28; 4th: P.20, Q30(i))
A number x is called an **accumulation point** of a set E if it is a point of closure of $E \setminus \{x\}$. Show that the set E' of accumulation points of E is a closed set.

Notation: E^c or E'

9. (3rd: P.46, Q29; 4th: P.20, Q30(ii))
Show that $\overline{E} = E \cup E'$.

10.* (3rd: P.46, Q30; 4th: P.20, Q31)
A set E is called **isolated** if $E \cap E' = \phi$. Show that every isolated set of real numbers is countable.

11*. Let $G \supseteq U \triangle E$. Show that $G \cup U \supseteq E$ (if $x \in$ RHS but $x \notin U$ then $x \in G$) and $(G \cup U) \setminus E \subseteq G$.