Recall:

Area of region $\overset{\Omega}{X(\mathcal{U})}$ where $X: \mathcal{U} \to S$ is a parametrization:

$$\text{Area}(\Omega) = \int_\Omega dA = \iint_{\mathcal{U}} \|X_u \times X_v\| \, du \, dv.$$

$$= \iint_{\mathcal{U}} \sqrt{EG - F^2} \, du dv$$

where $[g] = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$ is the 1st f.f. of S under parametrization X.

~~ ~~ last lecture: the defn is indep. of parametrization

$$\text{Area}(Y(V)) = \text{Area}(\Omega) = \text{Area}(X(\mathcal{U}))$$

$$\iint_V \sqrt{\tilde{E}\tilde{G} - \tilde{F}^2} \, d\eta \, d\xi \qquad\qquad \iint_{\mathcal{U}} \sqrt{EG - F^2} \, du dv.$$

Eg: find $\text{Area}(\mathbb{S}^2) = 4\pi$ (from Basic trigo.)

Use parametrization $X: (0, 2\pi) \times (0, \pi) \to \mathbb{S}^2$ given by

$$X(\varphi, \theta) = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$$

$$\begin{cases} X_\varphi = (-\sin\theta \sin\varphi, \sin\theta \cos\varphi, 0) \\ X_\theta = (\cos\theta \cos\varphi, \cos\theta \sin\varphi, -\sin\theta) \end{cases}$$

$$\Rightarrow [g] = \begin{bmatrix} \sin^2\theta & 0 \\ 0 & 1 \end{bmatrix}$$

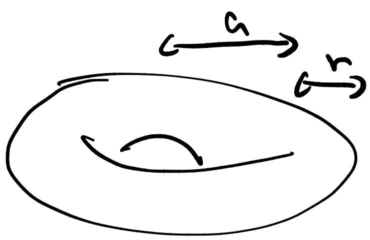
$$\Rightarrow \sqrt{\det[g]} = \sin\theta$$

$$\iint_U \sqrt{\det[g]}\, d\theta\, d\varphi = \int_0^{2\pi}\int_0^{\pi} \sin\theta \; d\theta\, d\varphi$$

$$= 2\pi \cdot \left[-\cos\theta\right]\Big|_0^{\pi}$$

$$= 2\pi\left(-\cos\pi + \cos 0\right) = 4\pi. \#$$

<u>Example</u>: surface area of Donut.



$X : (0, 2\pi) \times (0, 2\pi) \longrightarrow S$ given by

$$X(u,v) = \big((a + r\cos u)\cos v,\ (a + r\cos u)\sin v,\ r\sin u\big)$$

" X covers S except measure zero set."

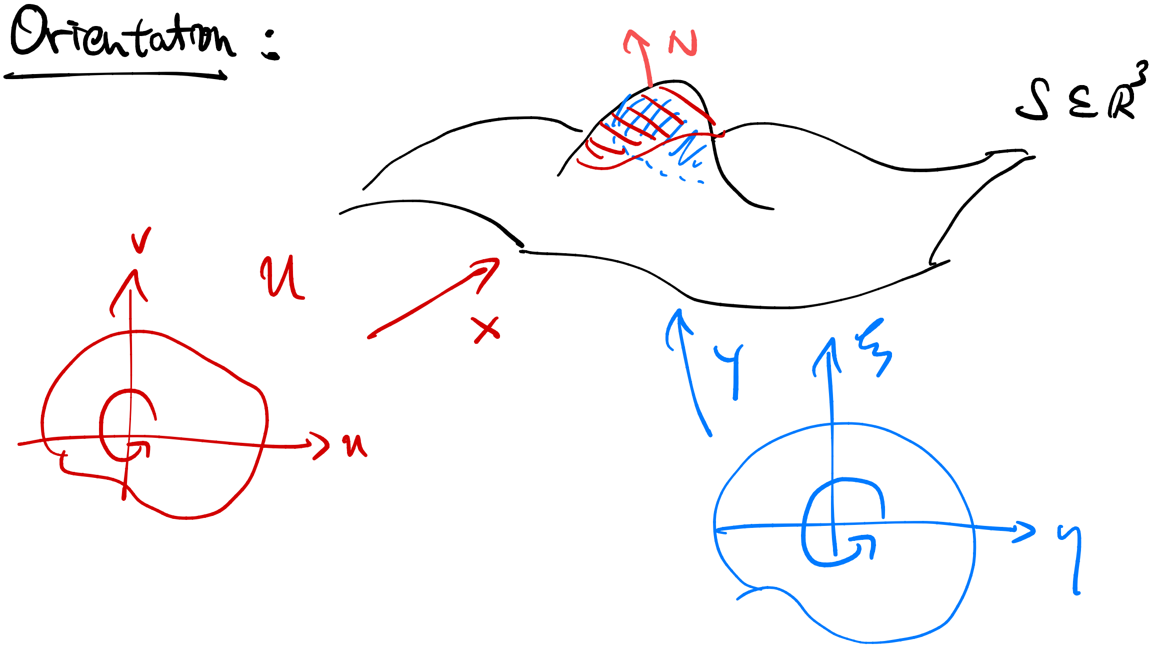
$$\begin{cases} X_u = (-r\sin u\cos v,\ -r\sin u\sin v,\ r\cos u) \\ X_v = (-(a + r\cos u)\sin v,\ (a + r\cos u)\cos v,\ 0) \end{cases}$$

$$\Rightarrow [g] = \begin{bmatrix} r^2 & 0 \\ 0 & (a + r\cos u)^2 \end{bmatrix}$$

$$\Rightarrow \sqrt{\det[g]} = r(a + r\cos u)$$

$$\iint_U r(a + r\cos u)\, du\, dv = 2\pi r \cdot \int_0^{2\pi} a + \cancel{r\cos u}\, du = 4\pi^2 r a \#.$$

Orientation:



finding the normal:

$$N_X = \frac{X_u \times X_v}{\| X_u \times X_v \|} \quad \text{v.s.} \quad \frac{Y_\eta \times Y_\xi}{\| Y_\eta \times Y_\xi \|} = N_Y$$

from "change of coordinate":

$$\begin{cases} Y_\eta = \dfrac{\partial u}{\partial \eta} X_u + \dfrac{\partial v}{\partial \eta} X_v \\[2ex] Y_\xi = \dfrac{\partial u}{\partial \xi} X_u + \dfrac{\partial v}{\partial \xi} X_v . \end{cases}$$

$$\Rightarrow \quad Y_\eta \times Y_\xi = \left(\frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi} - \frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta} \right) X_u \times X_v .$$

$$= \det\left(\frac{\partial(u,v)}{\partial(\eta,\xi)} \right) X_u \times X_v$$

$$\left(\text{Equivalently,} \quad \det[g]_Y = \det\left[\frac{\partial(u,v)}{\partial(\eta,\xi)} \right] \cdot \det[g]_X \right).$$

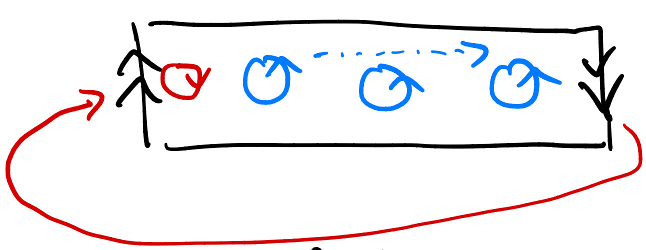
$\therefore \quad N_x = N_y \quad \text{iff} \quad \det\left[\frac{\partial(u,v)}{\partial(y,\bar{s})}\right] > 0.$

Defn: A regular surface S is called orientable if S can be covered by family $\{X_\alpha\}$ of parametrization s.t. if $p \in X_\alpha(\mathcal{U}_\alpha) \cap X_\beta(\mathcal{U}_\beta)$, then the change of coordinate has the Jacobian at p. (i.e. $\det\left[\frac{\partial(u,v)}{\partial(y,\bar{s})}\right] > 0$).

Example: If $f : \mathcal{U} \to \mathbb{R}$ is differentiable, ($\mathcal{U} \subseteq \mathbb{R}^2$)
then $S = \{(x, y, f(x,y)) \mid (x,y) \in \mathcal{U}\}$

is orientable.

Example: Möbius strip is <u>non-orientable</u>.



How to characterize it??

(Not a rigorous pf!)

Recall: If such $\{X_\alpha\}$ exists, we can define

$$N = \frac{X_u \times X_v}{\| X_u \times X_v \|}$$

← indep. of choice of X.

i.e. $= \frac{Y_\eta \times Y_\xi}{\| Y_\eta \times Y_\xi \|}$

① differentiable.

② $N \perp T_pS$, $\forall p \in S$.

③ $\| N \| \equiv 1$, $\forall p \in S$.

$\Rightarrow$ $\exists$ a differentiable "vector field" of unit normal

$$N : S \to \mathbb{R}^3.$$

Prop: A regular surface is orientable iff.

$\exists$ differentiable $N : S \to \mathbb{R}^3$ s.t. $N \perp T_pS$, $\|N\| = 1$, $\forall p \in S$.

Rmk: In this way, N refers to an orientation of S.

pf: ($\Rightarrow$) done by above discussion.

($\Leftarrow$): Assume S has only one connected component (WLOG).

Suppose S is covered by $\{X_\alpha\}_{\alpha \in A}$,

on each $X_\alpha(\mathcal{U}_\alpha)$, define

$$f_\alpha = \left\langle N, \frac{X_u^\alpha \times X_v^\alpha}{\| X_u^\alpha \times X_v^\alpha \|} \right\rangle$$

$\Rightarrow$ f_α is differentiable and takes discrete value ± 1 only on $X_\alpha(\mathcal{U}_\alpha) \subseteq S$.

cts $\Rightarrow$ $f_\alpha \equiv 1$ on $X_\alpha(\mathcal{U}_\alpha)$ (WLOG).

$$\Rightarrow \quad N = \frac{X_u^\alpha \times X_v^\alpha}{\| X_u^\alpha \times X_v^\alpha \|} \quad \text{on } X_\alpha(\mathcal{U}_\alpha)$$

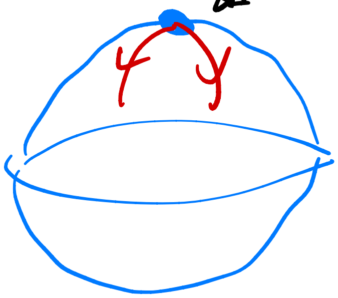
Repeat this for each X_α gives

$$\frac{X_u^\alpha \times X_v^\alpha}{\| X_u^\alpha \times X_u^\alpha \|} = \frac{X_{\bar{u}}^{\bar{\alpha}} \times X_{\bar{v}}^{\bar{\alpha}}}{\| X_{\bar{u}}^{\bar{\alpha}} \times X_{\bar{v}}^{\bar{\alpha}} \|}, \quad \forall \alpha, \bar{\alpha}$$

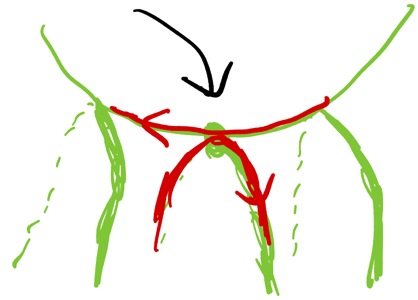
$$\Rightarrow \det\left(\frac{\partial(u,v)}{\partial(\bar{u},\bar{v})}\right) > 0 \quad \#.$$

Curvature of surface:

what means by curvature ??



V.S.



$\Downarrow$

Curvature CANNOT be a scalar quantity !!

<u>Defn</u>: let $S \subseteq \mathbb{R}^3$ be a regular surface with orientation N.
The shape operator S_p with respect to N at p is the operator given by: $\forall\, v \in T_pS$, let $\alpha : (-\varepsilon, \varepsilon) \to S$ be a differentiable curve on S with $\alpha(0) = p$, $\alpha'(0) = v$.

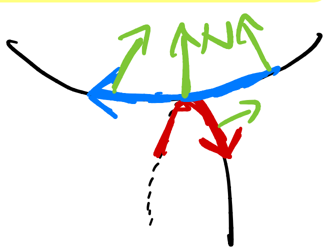
Then $S_p(v) \triangleq -\left.\dfrac{d}{dt}\right|_{t=0} N(\alpha(t))$.

convention only.

* Here, the map $N : S \to \mathbb{S}^2$ is called the Gauss map of S.

* $S_p = -dN_p : T_pS \to T_{N(p)}\mathbb{S}^2$, in term of differential of maps.

* Sometimes, ppl called S_p, the Weingarten map of S at p



* along red, blue, the change of N behave oppositely.

Example: Plane S : $ax+by+cz=d$.

$$N \equiv \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}} \Rightarrow N: S \to S^2 \text{ is a constant map}$$

$$\Rightarrow S_p = dN_p \equiv 0. \text{ ("no curvature")}$$

Example: $S = S^2$. $N = (x,y,z) \in S^2 = S$

$$(\text{or } \overline{N} = -(x,y,z).)$$

At $p = (x_0, y_0, z_0) \in S = S^2$.

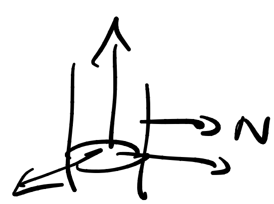
let $\alpha(t) = (x(t), y(t), z(t))$ where $\alpha(0) = p$, $\alpha'(0) = v$.

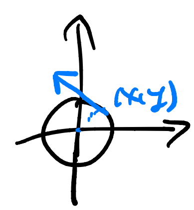
$$\Rightarrow N(\alpha(t)) = (x(t), y(t), z(t))$$

$$-\frac{d}{dt}\Big|_{t=0} N(\alpha(t)) = -(x'(0), y'(0), z'(0)) = -v.$$

$$\boxed{\therefore S_p = -Id.}$$

Example: cylinder $x^2+y^2=1$.





$$N(x,y,z) = (x,y,0).$$

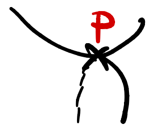
$$\Rightarrow -\frac{d}{dt}\Big|_{t=0} N(\alpha(t)) = -(x'(0), y'(0), 0)$$

If $\alpha'(0) = v \in T_pS$, then $v \in \text{span}\{\underset{W_1}{(0,0,1)}, \underset{W_2}{(-y,x,0)}\}$

where $p = (x,y,z)$.

$$\Rightarrow \begin{cases} S_p(w_1) = 0 \\ S_p(w_2) = -(x'(0), y'(0), 0) = (y_0, -x_0, 0) \\ \qquad\quad = -w_2. \end{cases}$$

Example :



$z = y^2 - x^2$

$$X(u,v) = (u, v, v^2 - u^2)$$

$$\begin{cases} X_u = (1, 0, -2u) \\ X_v = (0, 1, 2v) \end{cases} \Rightarrow N = \left(\frac{u}{\sqrt{u^2+v^2+1/4}}, \frac{-v}{\sqrt{u^2+v^2+1/4}}, \frac{1}{2\sqrt{u^2+v^2+1/4}} \right)$$

At $p = (0,0,0)$, $T_pS = \text{span}\{(1,0,0), (0,1,0)\}$

$$\begin{cases} dN_p(1,0,0) = \left.\frac{d}{dt}\right|_{t=0} N(t,0) = (2, 0, 0) \\ dN_p(0,1,0) = \left.\frac{d}{dt}\right|_{t=0} N(0,t) = (0,-2,0) \end{cases}$$

$\therefore$ eigenvector of S_p $= (1,0,0)$ and $(0,1,0)$

$\lambda_1 = -2 \qquad \lambda_2 = 2.$

Observation : Image of S_p ??

Prop : The shape operator S_p satisfies

① $S_p : T_pS \to T_pS$

② S_p is self-adjoint. (i.e. $\langle S_p(v), w \rangle = \langle v, S_p(w) \rangle$. $\forall v, w \in T_pS$.)

pf:

① : Since $\|N\| \equiv 1$

$$\Rightarrow \quad \left\langle N(\alpha(t)), \frac{d}{dt} N(\alpha(t)) \right\rangle \equiv 0, \ \forall t$$

$$\Rightarrow \quad \langle N_p, S_p(v) \rangle = 0, \quad \text{since } \alpha \text{ is arbitrary.}$$

$$\therefore \quad S_p \in T_pS = N_p^{\perp}$$

② let $v, w \in T_pS$. $\Rightarrow \begin{cases} v = a X_u + b X_v \\ w = c X_u + d X_v \end{cases}$

$$\begin{aligned} -\langle S_p(v), w \rangle &= \langle a N_u + b N_v, \ c X_u + d X_v \rangle \\ &= ac \langle N_u, X_u \rangle + bd \langle N_v, X_v \rangle \\ &\quad + ad \langle N_u, X_v \rangle + bc \langle N_v, X_u \rangle \end{aligned}$$

$$\begin{aligned} -\langle S_p(w), v \rangle &= \langle c N_u + d N_v, \ a X_u + b X_v \rangle \\ &= ac \langle N_u, X_u \rangle + bd \langle N_v, X_v \rangle \\ &\quad + cb \langle N_u, X_v \rangle + ad \langle N_v, X_u \rangle \end{aligned}$$

$$\langle N, X_u \rangle \equiv 0 \equiv \langle N, X_v \rangle$$

$$\Rightarrow \begin{cases} \langle N_v, X_u \rangle + \langle N, X_{uv} \rangle = 0 \\ \langle N_u, X_v \rangle + \langle N, X_{uv} \rangle = 0 \end{cases}$$

$$\Rightarrow \quad \langle N_u, X_v \rangle = \langle N_v, X_u \rangle.$$

#.

Self-adjoint

$\Rightarrow \quad (v,w) \longmapsto \langle S_p(v), w \rangle$ define a symmetric bilinear form on T_pS !!

<u>Defn</u>: $\mathrm{II}_p(v,w) \triangleq \langle S_p(v), w\rangle = g(S_p(v), w)$ is called the 2nd fundamental form of S at p.

This is symmetric since $S_p = S_p^*$ (self-adjoint)

determined by II_p on $\{X_u, X_v\}$

<u>Defn</u>: Given $N = \dfrac{X_u \times X_v}{\|X_u \times X_v\|}$, we define the coefficient of the 2nd f.f. at p to be

$$\begin{cases} e = \mathrm{II}_p(X_u, X_u) \\ f = \mathrm{II}_p(X_u, X_v) = \mathrm{II}_p(X_v, X_u) \\ g = \mathrm{II}_p(X_v, X_v) \end{cases}$$

Q: formula for e, f, g ??