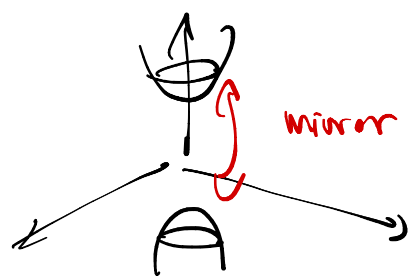
Recap:

prop: If $F: \mathcal{U} \in \mathbb{R}^3 \to \mathbb{R}$ is differentiable on $\mathcal{U}$ and $c \in \mathbb{R}$ is s.t. $\forall p \in F^{-1}(c)$, $\nabla F(p) \neq 0$,

then $F^{-1}(c)$ is a regular surface.

eg: $F(x,y,z) = -x^2 - y^2 + z^2 - 1$

$F^{-1}(0) =$ hyperboloid



$\nabla F = (-2x, -2y, 2z)$ s.t. if $F(x,y,z) = 0$

then $(x,y,z) \neq 0$

$S = F^{-1}(0)$ is regular surface. $\Leftarrow$ $\nabla F \neq 0$

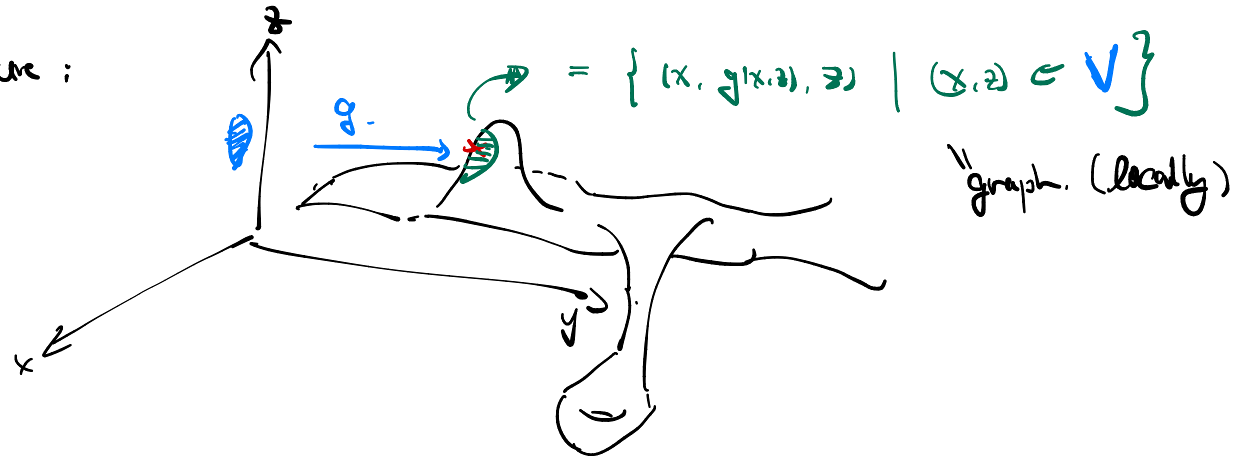
But S is Not connected. This does Not ensure connectedness.

☆ Graphic $\Rightarrow$ regular surface   Q: converse ??

prop: Let $S \subseteq \mathbb{R}^3$ be a regular surface, $p \in S$.

Then $\exists\, \mathcal{U}$ nbd of p in S s.t. $\mathcal{U} =$ graph of a differentiable fcn which has one of the following forms: $z = f(x,y)$, $y = g(x,z)$ or $x = h(y,z)$.

picture :



pf: let $X : V^{\subseteq \mathbb{R}^2} \to S$ be a local parametrization of S around $p \in S$. s.t.

$$X(u,v) = (x(u,v), y(u,v), z(u,v)).$$

dX = full rank at $q \in X^{-1}(p)$

$$\Rightarrow \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix} = \text{rank } 2. \text{ at } X^{-1}(p) = q \in V.$$

WLOG, assume $\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix}$ = rank 2. at q. (ie det $\neq 0$)

Then $\underset{\tilde{X}}{\pi \circ X(u,v)}$ where $\pi(x,y,z) = (x,y)$ satisfies $\tilde{X} : V \to \mathbb{R}^2$ given by

$$\tilde{X}(u,v) = (x(u,v), y(u,v))$$

Inverse fcn thm $\Rightarrow \exists\, \mathcal{U}_1$, open nbd of q and $\mathcal{U}_2$ open nbd of $\pi(p)$ s.t. $\tilde{X}|_{\mathcal{U}_1} : \mathcal{U}_1 \to \mathcal{U}_2$ is a diffeomorphism.

then on U_2,

$$(x,y) \mapsto (u(x,y), v(x,y)) \text{ is differentiable..}$$

s.t.

$$z = z(u,v) = z\big(u(x,y), v(x,y)\big) = f(x,y).$$

#.

Eg: $S = \{ (x,y,z) \mid z = \sqrt{x^2+y^2} \}$ is

Not a regular surface.

☆☆ there might be some good way of parametrization.

pf: If yes, $\exists\, \mathcal{U}$ open nbd of $(0,0,0)$ s.t. it is a graph in form of either

$$\underline{z = f(x,y)}, \quad \underline{y = g(x,z)} \quad \text{or} \quad \underline{x = h(y,z)}.$$

$$z_0 = \sqrt{x_0^2 + \boxed{y_0^2}} = \sqrt{x_0^2 + \boxed{(-y_0)^2}}$$

$\Rightarrow S$ cannot be graph over (x,z) or (y,z).

$$\Downarrow$$

$z = \sqrt{x^2+y^2}$ $\neq$ differentiable at 0.

$\to\leftarrow$ #

(From the proof).

prop: let $p \in S$ where S is a regular surface,

let $X : \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a map s.t. $p \in X(\mathcal{U}) \subseteq S$.

If ① X is smooth

② $dX(q)$ = full rank $\forall q \in \mathcal{U}$.

③ X = 1-1

then X^{-1} is cts. (s.t. X is homeomorphism $\Rightarrow$ local parametrization)

pf:

Suppose $X(u,v) = (x(u,v), y(u,v), z(u,v))$

let $q \in \mathcal{U}$. s.t. $\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix}$ = rank 2 at q.

WLOG assume $\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix}$ = rank 2 at q. s.t.

$\overset{\text{IFT}}{\Longrightarrow}$ $\pi \circ X = (x(u,v), y(u,v))$ is a diffeomorphism around q.

i.e. $\exists\, \mathcal{U}_1 \ni q, \mathcal{U}_2$ s.t. $\pi \circ X : \mathcal{U}_1 \to \mathcal{U}_2$ is a diff..

$\Rightarrow$ $X^{-1}(x,y,z) = (\pi \circ X)^{-1}(x,y) = (u,v)$ is differentiable at p

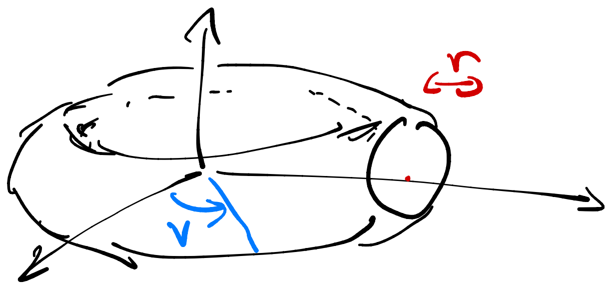
and hence is cts.

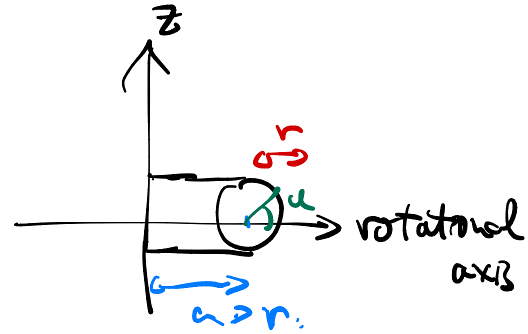
☆☆ suffice to check ① regularity ② 1-1 at differentiable level
③ 1-1 at topo. level !!

Eg: Torus (Donut).

$$X(u,v) = \big((r\cos u + a)\cos v,\ (r\cos u + a)\sin v,\ r\sin u\big)$$

where $u \in (0, 2\pi)$, $v \in (0, 2\pi)$





① X is smooth ✓

② $X_u = (-r\sin u\cos v,\ -r\sin u\sin v,\ r\cos u)$

$X_v = (-(r\cos u + a)\sin v,\ (r\cos u + a)\cos v,\ 0)$

If $\cos u \neq 0$, $\Rightarrow$ linearly indep.

If $\cos u = 0 \Rightarrow \sin u = \pm 1$.

$$\Rightarrow \begin{cases} X_u = \pm r(\cos v,\ \sin v,\ 0) \\ X_v = a(-\sin v,\ \cos v,\ 0) \end{cases}$$

$\leftarrow$ linearly indep. by $\det \neq 0$.

③ 1-1: X is 1-1.

If $X(u,v) = X(\tilde{u}, \tilde{v}) = (x, y, z)$

$$\Rightarrow \begin{cases} \sin u = \sin \tilde{u} = z \\ x^2 + y^2 = (r\cos u + a)^2 = (r\cos\tilde{u} + a)^2. \end{cases}$$

$\Rightarrow u = \tilde{u}$ since otherwise $\cos u = -\cos \tilde{u}$.

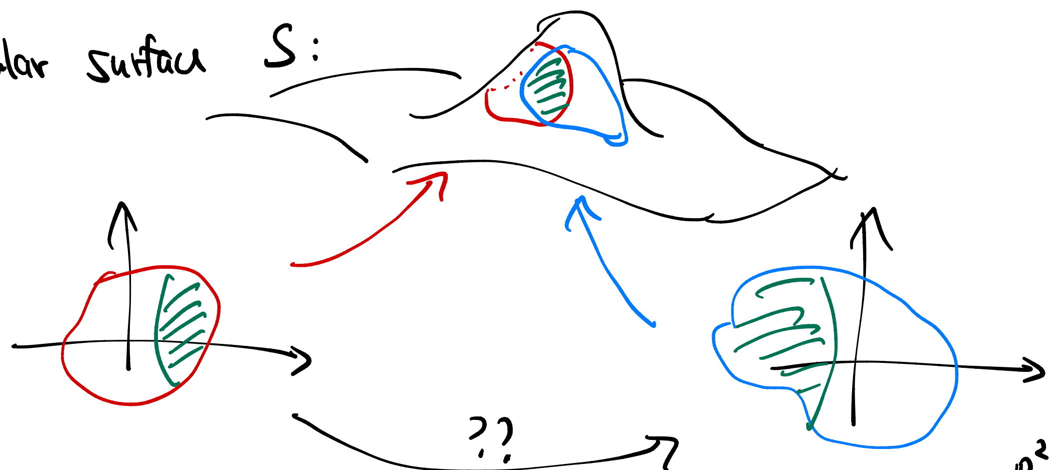
$\Rightarrow \cos u = 0$, $\sin u = 1$

$\Rightarrow x = y = 0 \rightarrow\leftarrow$.

Then $\begin{cases} \cos v = \cos \tilde{v} \\ \sin v = \sin \tilde{v} \end{cases} \Rightarrow v = \tilde{v}$ #.

By considering x, y.

Regular surface S:



prop: Let $p \in S$, a regular surface. and let $X : \mathcal{U} \to S$, $\mathcal{U} \subseteq \mathbb{R}^2$,

$Y : V \to S$, $V \subseteq \mathbb{R}^2$ be two parametrization of S around $p \in S$ s.t. $p \in X(\mathcal{U}) \cap Y(V) = W$.

Then the change of coordinate $h = X^{-1} \circ Y : Y^{-1}(W) \to X^{-1}(W)$ is a diffeomorphism.

pf: By earlier prop, $\exists\, S_1 \subset X(\mathcal{U}) \cap Y(V)$ s.t.

S_1 is given by graph $= \{ (x, y, f(x,y)) \mid (x,y) \in \mathcal{O} \}$

where $\mathcal{O}$ is some open set in $\mathbb{R}^2$.

$\Rightarrow X(u,v) = (x(u,v),\ y(u,v),\ f(x(u,v), y(u,v)))$ there

$$\Rightarrow \begin{cases} X_u = (x_u,\ y_u,\ \boxed{f_x x_u + f_y y_u}) \\ X_v = (x_v,\ y_v,\ \boxed{f_x x_v + f_y y_v}) \end{cases}$$

linearly combination of first 2 columns.

$\Rightarrow$ (x_u, y_u), (x_v, y_v) are linearly indep.

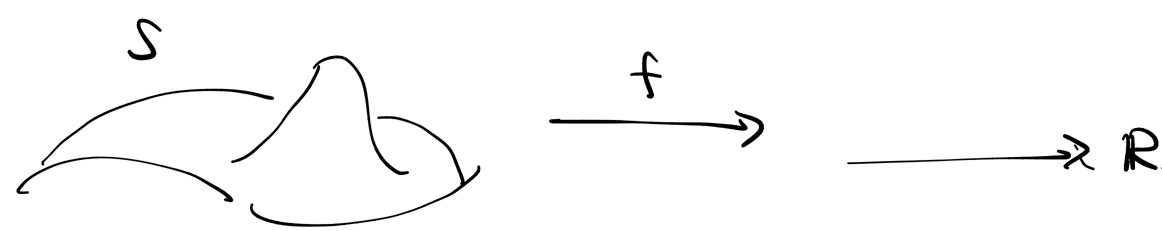
$\Rightarrow$ $(u,v) \mapsto (x,y)$ is diffeomorphism around $X^{-1}(p)$

local coord. of Y.

Similarly $\Rightarrow$ $(\xi, \eta) \mapsto (x,y)$ is diff. around $Y^{-1}(p)$.

$\Rightarrow$ $(u,v) \mapsto (\xi, \eta)$ is a diff. #.

☆



given $f: S \to \mathbb{R}$, a fcn on the regular surface.

Q: differentiability of f at $p \in S$??

Ans: Define it using parametrization

Def: $f: V \subseteq S \to \mathbb{R}$ is a function defined on open subset in S. Then f is differentiable at $p \in V$ if $\exists$ parametrization $X: U \to S$ with $p \in X(U) \subset V$ s.t.

$f \circ X : \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $X^{-1}(p)$.

☆ f is differentiable on V if f is diff at all pt in V.

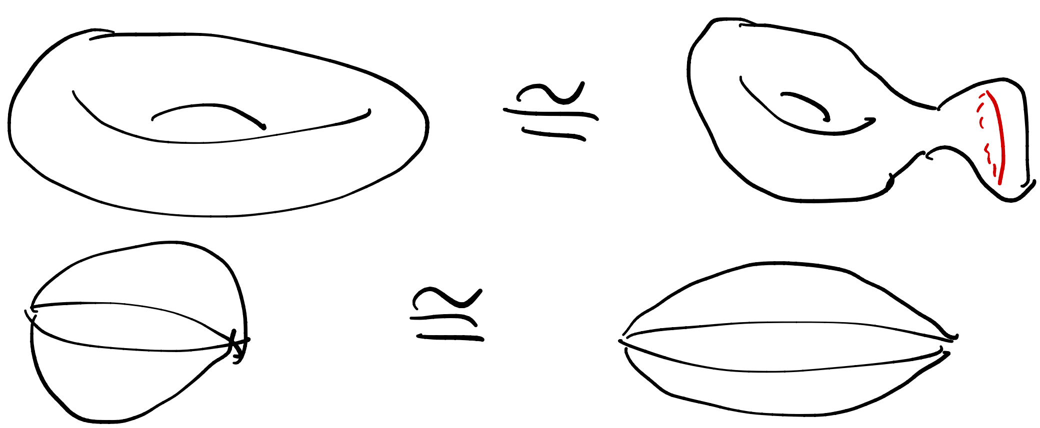
Extend it to maps between surface naturally !!

Defn: Let M, N be regular surface, $F : M \to N$ be a map. F is said to be differentiable (smooth) iff $\forall p \in M$, local parametrization $X : \mathcal{U} \to M$ around p, and $q = F(p) \in N$, local parametrization $Y : V \to N$ around q, $Y^{-1} \circ F \circ X$ is differentiable wherever it is defined.

Defn: Two surface M, N are diffeomorphic if $\exists$ differentiable map $\varphi : M \to N$ s.t. it has a differentiable inverse $\varphi^{-1} : N \to M$.

picture :

① ②



$S = \{x^2+y^2+z^2=1\}$ $\tilde{S} = \{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\}$

③ regular surface are locally diffeomorphi

to open set in $\mathbb{R}^2$.