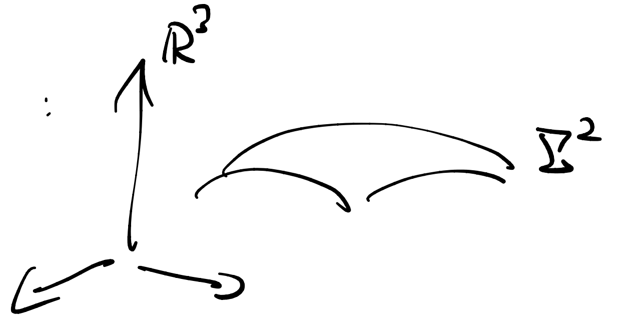
Week 3: Surfaces

Interested in 2D objects:



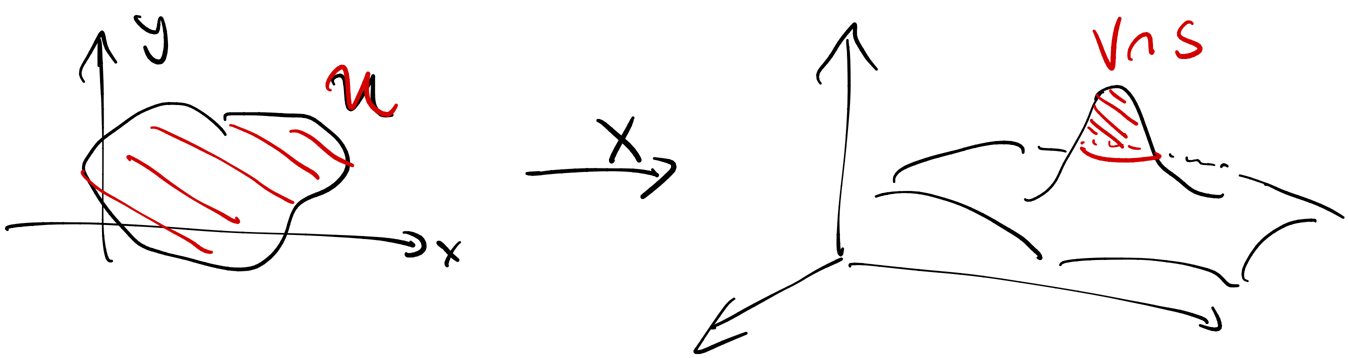
nice 2-dim objects: regular surfaces

(no sharp points, no edge, no self intersections.)

Mathematically:

Defn: A subset $S \subseteq \mathbb{R}^3$ is a regular surface if $\forall$ $p \in S$, $\exists$ open nbd $V \subseteq \mathbb{R}^3$, open set $U \subseteq \mathbb{R}^2$, and a map $X: U \to V \cap S$ s.t.

1. X is smooth;
2. X is a homeomorphism;
3. $\forall\, q \in U$, $dX_q: \mathbb{R}^2 \to \mathbb{R}^3$ is injective.



Here, the mapping X is called parametrization

or system of local coord.

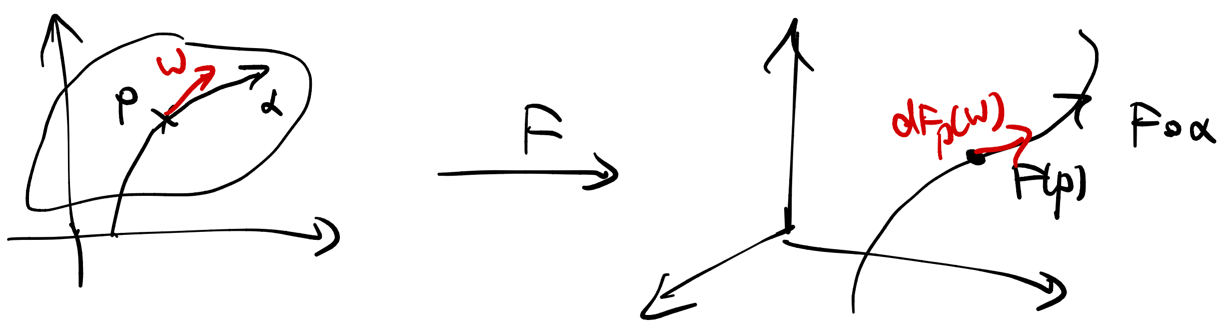
* i.e. A regular surface in $\mathbb{R}^3$ is a set $\subseteq \mathbb{R}^3$ which can be covered by family of coordinate charts.

<u>Defn</u> (differential of smooth map)

let $F: U \overset{\text{open}}{\subseteq} \mathbb{R}^n \to \mathbb{R}^m$. $\forall p \in U$, we define a linear map $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ (called differential of F at p)

as $dF_p(w) = \frac{d}{dt}\Big|_{t=0} F \circ \alpha(t)$ where $\alpha : (-\varepsilon, \varepsilon) \to U$

is a differentiable curve s.t. $\begin{cases} \alpha(0) = p \\ \alpha'(0) = w. \end{cases}$



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prop : The defn of dF_p is independent of choice of α. Hence, it is well-defined.

pf: W.L.O.G, assume $n=2$, $m=3$ (the general case is similar)

let $F(x,y) = (f_1, f_2, f_3) \in \mathbb{R}^3$.

WLOG, assume $p = (0,0)$

let $w = (a,b) \in \mathbb{R}^2$, $\alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$

where $\begin{cases} \alpha(0) = (0,0) \\ \alpha'(0) = (a,b). \end{cases}$, $\alpha(t) = (x(t), y(t))$

Then

$$F \circ \alpha(t) = \Big(f_1(x(t), y(t)),\ f_2(x(t), y(t)),\ f_3(x(t), y(t)) \Big)^T$$

$$\frac{d}{dt}\Big|_{t=0} F\circ\alpha = \left(\frac{\partial f_i}{\partial x} \cdot \frac{\partial x}{\partial t}\Big|_0 + \frac{\partial f_i}{\partial y} \frac{\partial y}{\partial t}\Big|_0 \right)^T$$

$$= \left(\frac{\partial f_i}{\partial x}\Big|_p \cdot a + \frac{\partial f_i}{\partial y}\Big|_p \cdot b \right)^T$$

$$\Rightarrow \quad dF_p = \begin{bmatrix} \frac{\partial f_1}{\partial x}\Big|_p & \frac{\partial f_1}{\partial y}\Big|_p \\ \frac{\partial f_2}{\partial x}\Big|_p & \frac{\partial f_2}{\partial y}\Big|_p \\ \frac{\partial f_3}{\partial x}\Big|_p & \frac{\partial f_3}{\partial y}\Big|_p \end{bmatrix} \quad \#$$

$$\begin{array}{cc} \| & \| \\ dF_p(e_1) & dF_p(e_2) \end{array}$$

where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

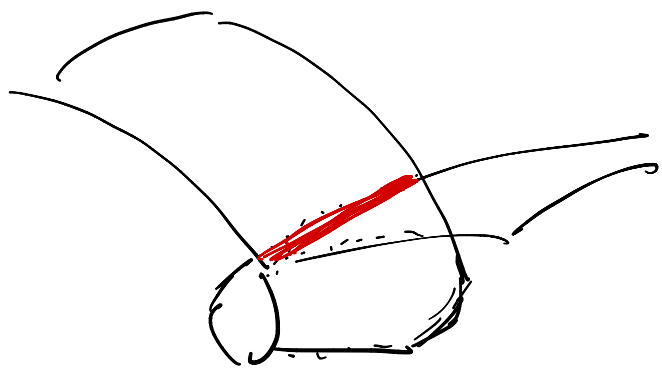
#: $dX_p = 1\text{-}1$ in def of surface:

rank of matrix $= 2$

$\Longleftrightarrow \{ dX_p(e_1), dX_p(e_2) \}$ are linearly indep.

$\star$: X = bijection $\Rightarrow$ prevent self - intersection

counter-example



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prop : If $f: \mathcal{U} \to \mathbb{R}$ is a differentiable fcn on open subset $\mathcal{U} \subseteq \mathbb{R}^2$. Then $S = \{ (x, y, f(x,y)) \mid (x,y) \in \mathcal{U} \}$ is a regular surface.

pf: clearly, $X: \mathcal{U} \to V$ given by

$V \subseteq \mathcal{U} \times \mathbb{R} \subseteq \mathbb{R}^3$

① $X(u,v) = (u, v, f(u,v))$ is smooth.

② $\begin{cases} dX_p(e_1) = (1, 0, f_u) \\ dX_p(e_2) = (0, 1, f_v) \end{cases}$ linearly indep.

③ $X^{-1} : V \to U$ given by

$$X^{-1}(u, v, f(u,v)) = (u,v)$$ is cts. #

<u>Example</u>: $\mathbb{S}^2 \subseteq \mathbb{R}^3$

$$\mathbb{S}^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$$

local charts:

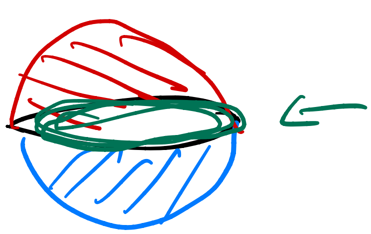
- $X_1 : U_1 \to \mathbb{R}^3$ where

$$U_1 = \{(x,y) \mid x^2+y^2 < 1\}$$

$$X_1(u,v) = (u, v, \sqrt{1-u^2-v^2})$$

- $X_2 : U_1 \to \mathbb{R}^3$ by

$$X_2(u,v) = (u, v, -\sqrt{1-u^2-v^2})$$



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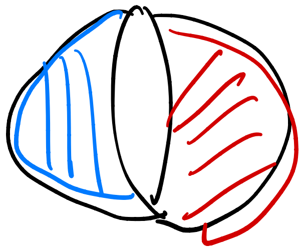

$$X_3(u,v) = (u, \sqrt{1-u^2-v^2}, v)$$

$$X_4(u,v) = (u, -\sqrt{1-u^2-v^2}, v)$$

$$X_5 = (\sqrt{1-u^2-v^2}, u, v)$$

$$X_6 = (-\sqrt{1-u^2-v^2}, u, v)$$

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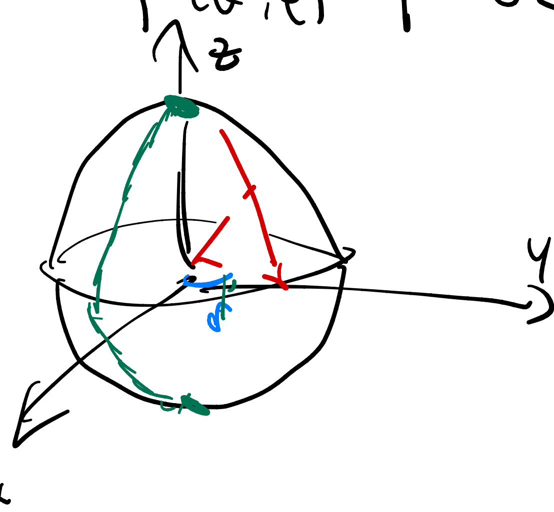
other coordination of S^2

Ⓐ spherical coordinate

$$X(\theta, \varphi) = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$$

$$U = \{(\theta, \varphi) \mid 0 < \theta < \pi, \ 0 < \varphi < 2\pi\}$$



green part $\| \ \{x \geq 0, y = 0\}$

Checking : local chart

① differentiable ✓

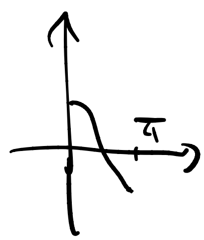
②
$$\begin{cases} \dfrac{\partial x}{\partial \theta} = (\cos\theta\cos\varphi,\ \cos\theta\sin\varphi,\ -\sin\theta) \\[2ex] \dfrac{\partial x}{\partial \varphi} = (-\sin\theta\sin\varphi,\ \sin\theta\cos\varphi,\ 0) \end{cases}$$

is linearly indep unless $\sin\theta = 0 \Rightarrow \theta \in \pi\mathbb{Z}$.

$\Rightarrow\Leftarrow$.

③ given $(x, y, z) \in S^2 \setminus \{x \geq 0,\ y = 0\}$

take $\theta \in [0, \pi]$ s.t. $\cos\theta = z$. $\because |z| \leq 1$



if $\theta = 0$ or π,

$|z| = 1$

$\Rightarrow\ x = y = 0 \quad \Rightarrow\Leftarrow$

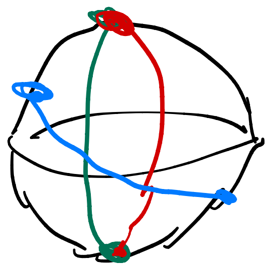
$\therefore\ \theta \in (0, \pi)$ and is unique.

($\theta = \cos^{-1} z$ is differentiable here)

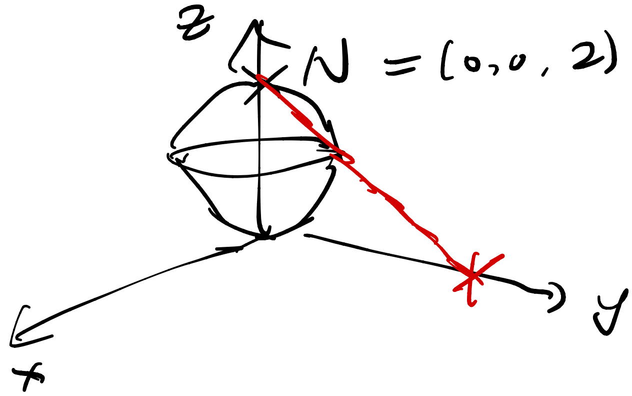
Now
$$\begin{cases} y = \sin\theta\cos\varphi \\ x = \sin\theta\sin\varphi \end{cases}$$
determine φ uniquely by $x, y, \theta(z)$.

#.

Construct X_2, X_3 by (diff.) rotation.



Stereo-graphic projection:



why??

$$X(u,v) = \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{2(u^2+v^2)}{u^2+v^2+4} \right)$$

$$X : \mathbb{R}^2 \longrightarrow S^2 \setminus N.$$

But $S^2 = \{ (x,y,z) \mid f(x,y,z) = 0 \}$

where $f(x,y,z) = x^2+y^2+z^2-1$

* Level set of a very nice fcn !!

Q: If $S = \{ (x,y,z) \mid F(x,y,z) = 0 \}$ for some smooth fcn $F: \mathbb{R}^3 \to \mathbb{R}$, is S always a regular surface ??

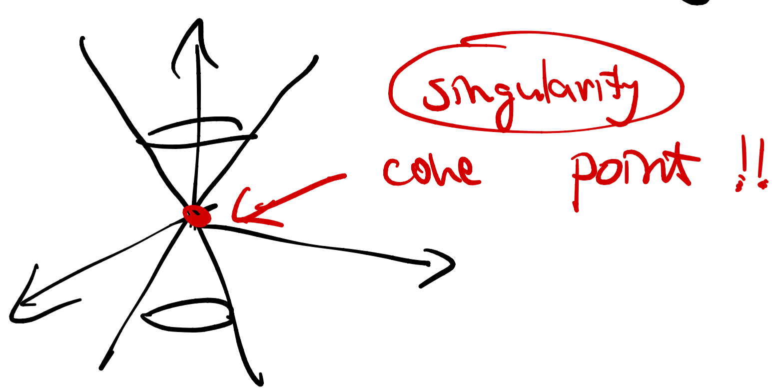
"Counter-example":

① $S = \{ (x,y,z) \mid x^2+y^2+z^2 = 0 \}$

then $S = \{(0,0,0)\} \neq$ surface.

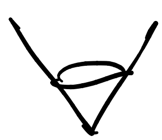
② $S = \{ (x,y,z) \mid \underbrace{x^2+y^2-z^2}_{F(x,y,z)} = 0 \}$

then S:



More, ...

from eg 2:



$S^+ = \{ F=0,\ z \geq 0 \}$

$= \{ (x,y,z) \mid z = \sqrt{x^2+y^2} \}$

viewed as graph, then $z = f(x,y)$

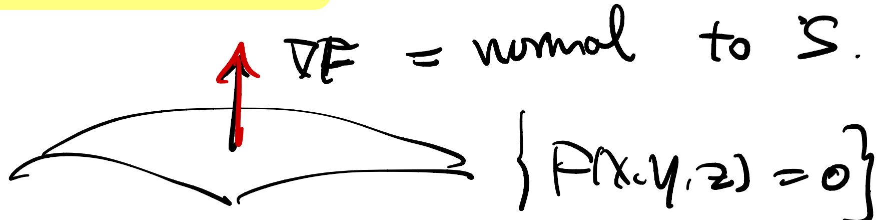
- $\dfrac{\partial f}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}}$, $\dfrac{\partial f}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}$

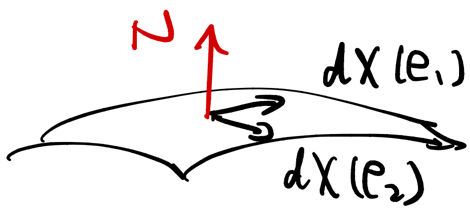
singular at $(0,0)$

- $\nabla F = (2x, 2y, -2z)$ $= 0$ at $(0,0,0)$!!

failure ?!

geometrically,

- 

- 

$N = dX(e_1) \times dX(e_2)$

is well defined when

dX = full rank.

prop: let $U \subseteq \mathbb{R}^3$ be open set.

let $F : U \to \mathbb{R}$ be differentiable fcn.

If $a \in \mathbb{R}$ is a regular value of F

$\left(\text{i.e. } \forall \underset{\parallel}{\overset{p}{(x,y,z)}} \text{ s.t. } F(x,y,z) = a, \text{ we have } dF(p) \neq 0 \Leftrightarrow dF(p) = \text{full rank} \right)$

then $S = F^{-1}(a)$ is a regular surface.

<u>pf:</u> let $p_0 \in S$ (i.e. $F(p_0) = a$)

$$\nabla F(p_0) = (F_x, F_y, F_z)\big|_{p_0} \neq 0$$

assume $F_z(p_0) \neq 0$.

Define $G(x,y,z) = (x, y, F(x,y,z))$ s.t.

$$dG = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ F_x & F_y & F_z \end{bmatrix} \Rightarrow \det(dG|_{p_0}) \neq 0$$

IFT $\Rightarrow \exists$ nbd V of p_0,

nbd $\mathcal{U}$ of $G(p_0)$ s.t.

G has a C^∞ inverse on $\mathcal{U}$.

$$G^{-1}(u,v,w) = (u, v, \tilde{F}(u,v,w))$$

$\boxed{\text{Recall : looking for } \mathbf{X}(u,v) \in S}$

taking $X : \mathcal{U} \to \mathbb{R}^3$ given by

$$X(u,v) = G^{-1}(u,v,a)$$
$$= (u,v,\ \tilde{F}(u,v,a)) \text{ s.t.}$$

Since

$$(u,v,a) = G \circ G^{-1}(u,v,a)$$
$$= G(u,v,\ \tilde{F}(u,v,a))$$
$$= (u,v,\ F(u,v,\tilde{F}))$$

$\Rightarrow\ F(X(u,v)) = a \quad (\text{i.e. } X(u,v) \in S)$

$\therefore$ S is locally a graph over $\mathcal{U}$.

$\Rightarrow$ regular surface!!