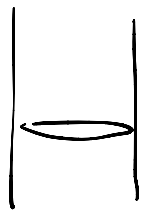
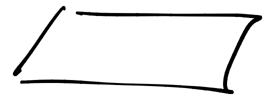
Intrinsic geometry :

Recall :



v.s.



cylinder

plane

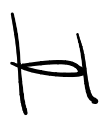
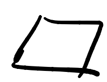
$X(u,v) = (\cos u, \sin u, v)$ $\qquad$ $X(u,v) = (u, v, 0)$

$\begin{cases} X_u = (-\sin u, \cos u, 0) \\ X_v = (0, 0, 1) \end{cases}$ $\qquad$ $\begin{cases} X_u = (1, 0, 0) \\ X_v = (0, 1, 0) \end{cases}$

$$\Downarrow \qquad \swarrow$$

* $[g]_X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (SAME 1st. f.f.)

$\Downarrow$

* But  $\neq$ . What common geometric structure??

Intrinsic geometry : properties of M only depending on $[g]$.

Extrinsic geometry : properties of M depends on how M's sit inside $\mathbb{R}^3$.

Eg: Mean curvature.

Q: what about Gaussian curvature K ??

$$K(p) = \det([S_p]) = \frac{\det[\mathrm{II}]_X}{\det[g]_X} = \frac{eg - f^2}{EG - F^2}.$$

<u>Thm</u> (Theorema Egregium of Gauss)
The Gaussian curvature K is invariant under isometries.
(Hence, K is a intrinsic information.)

<u>Defn</u>: $M, \tilde{M}$ = regular surface

Recall A diffeomorphism $\varphi : M \to \tilde{M}$ is a homeomorphism s.t. φ is differentiable and $d\varphi$ = full rank at each $p \in M$.

A isometry $\varphi : M \to \tilde{M}$ is a diffeomorphism s.t. $\forall p \in M$, $\forall u, v \in T_pM$,

$$g\big|_p(u,v) = \tilde{g}\big|_{\varphi(p)}(d\varphi(u), d\varphi(v))$$ (preserve inner product)

where $d\varphi(u), d\varphi(v) \in T_{\varphi(p)}\tilde{M}$, $\tilde{g}$ = inner product on $\tilde{M}$

In this case, $M, \tilde{M}$ are said to be isometric.

(Some abstract concept)

<u>Defn</u>: Given a surface M with inner product g.
$\varphi : \tilde{M} \to M$ is a diffeomorphism for some surface $\tilde{M}$,
We might define an inner product $\tilde{g}$ on $\tilde{M}$ by

$$\tilde{g}\big|_p(u,v) = g\big|_{\varphi(p)}(d\varphi(u), d\varphi(v)), \forall p \in \tilde{M}.$$

We will denote it as $\tilde{g} = \varphi^* g$ (pull-back of g)
(therefore $g = \varphi^* \langle , \rangle|_{\mathbb{R}^3}$).

Lemma φ preserve length $\Leftrightarrow$ φ preserves inner product.

i.e. $\|d\varphi(u)\|_{\tilde{g}} = \|u\|_g \quad \forall u \in T_pM$

iff $\langle d\varphi(u), d\varphi(v)\rangle_{\tilde{g}} = \langle u, v\rangle_g \quad , \forall u, v \in T_pM$

pf: $(\Leftarrow)$: trivial

$(\Rightarrow)$ Let $u(t) = u + tv$, $v \in T_pM$.

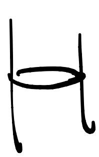
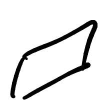
$$\langle \underbrace{d\varphi}_{\text{linear map}}(u_t), d\varphi(u_t)\rangle = \langle u_t, u_t\rangle \quad \forall t \in \mathbb{R}.$$

$$\Rightarrow \begin{cases} \left.\frac{d}{dt}\right|_{t=0} \langle d\varphi(u_t), d\varphi(u_t)\rangle = 2\langle d\varphi(u), d\varphi(v)\rangle_{\tilde{g}} \\ \qquad \parallel \\ \left.\frac{d}{dt}\right|_{t=0} \langle u_t, u_t\rangle = 2\langle u, v\rangle_g \end{cases} \qquad \#.$$

(Local Isometry)

Defn: A map $\varphi : V \to \tilde{S}$ ($V \subset S$) on a open nbd V of $p \in S$ is a local isometry at p if $\exists\ \tilde{V}$, open nbd of $\varphi(p)$ s.t. $\varphi : V \to \tilde{V}$ is an isometry.

Example :  ,  are isometric locally !!

But  $\neq$  topological $\Rightarrow$ Not globally isometric.

prop: Assume $X: \mathcal{U} \to M$, $\tilde{X}: \tilde{\mathcal{U}} \to \tilde{M}$ be two parametrized surface s.t. $g_{ij} = \tilde{g}_{ij}$ on $\mathcal{U}$. $\hat{\mathcal{U}}$.

Then $\varphi = \tilde{X} \circ X^{-1} : X(\mathcal{U}) \to \tilde{M}$ is an local isometry.

pf: suffices to show $\varphi^{*}\tilde{g} = g$ on $X(\mathcal{U})$.

at $p \in X(\mathcal{U})$, $w \in T_pM$

$$\Rightarrow \quad w = a X_u + b X_v \quad \text{at } p$$

$$\Rightarrow \quad \|w\|^2 = a^2 g_{uu} + 2ab\, g_{uv} + b^2 g_{vv}.$$

On the other hand, $d\varphi|_p(w) = \frac{d}{dt}\Big|_{t=0} \varphi \circ \alpha(t)$

where $\alpha: (-\varepsilon, \varepsilon) \to M$. is a regular curve.

$$\begin{tikzcd}
 & M \arrow[rr, "\varphi"] & & \tilde{M} & \\
\mathcal{U} \arrow[ur, "X"] & & & & \tilde{\mathcal{U}} \arrow[ul, "\tilde{X}"']
\end{tikzcd}$$

· $\alpha(t) = X \circ \beta(t)$ where $\beta'(0) = (a, b)$

since $\alpha'(0) = a X_u + b X_v$.

· $\tilde{X}^{-1} \circ \varphi \circ \alpha(t) = \tilde{X}^{-1} \circ \varphi \circ X \circ \beta(t)$

$$= \tilde{X}^{-1} \circ (\tilde{X} \circ X^{-1}) \circ X \circ \beta(t) = \beta(t).$$

$\therefore \quad d\varphi_p(w) = a\tilde{X}_u + b\tilde{X}_v$ at $\varphi(p) \in \tilde{M}$.

$\Rightarrow \quad \|d\varphi_p(w)\|^2 = a^2 \tilde{g}_{uu} + 2ab\,\tilde{g}_{uv} + b^2 \tilde{g}_{vv}$

$= \|w\|^2$ #

$\because$ w is arbitrary, $\varphi^*\tilde{g} = g$ #.

Example: Helicoid and Catenoid are locally isometric.

Catenoid: $X(u,v) = (a\cosh(v)\cos u,\ a\cosh(v)\sin u,\ av)$

$u \in (0, 2\pi)$, $v \in \mathbb{R}$.

$\Rightarrow \quad [g]_X = a^2\cosh^2(v)\cdot Id.$

Helicoid: $X(u,v) = (a\sinh(v)\cos u,\ a\sinh(v)\sin u,\ au)$

$u \in (0, 2\pi)$, $v \in \mathbb{R}$.

$\Rightarrow \quad [g]_X = a^2\cosh^2(v)\cdot Id$

Isometric locally.

Proof of Gauss Theorema Egregium:

Setting: $\cdot\ X: U \to M$.

$\cdot\ N = \dfrac{X_u \times X_v}{\|X_u \times X_v\|}$.

on nbd of $p \in M$, we have frame $\{X_u, X_v, N\}$

(using (i,j) notation, the computation below also works in $M^n \subset \mathbb{R}^{n+1}$)

$g_{ij} = \langle X_i, X_j \rangle$, $(g^{ij}) =$ inverse matrix of (g_{ij}).

(i.e. $\sum_{j=1}^{n} g^{ij} g_{jk} = \delta^i_k$)

$X_{ij} = \partial_i \partial_j X$ (as a vector in $\mathbb{R}^3$)

$\triangleq \Gamma^k_{ij} X_k + h_{ij} N$ for some (Γ^k_{ij}) coefficient.

(**Defn**: Γ^k_{ij} is called the Christoffel symbol wrt parametrization $X: U \to M$.)

Lemma: • $\Gamma^k_{ij} = \Gamma^k_{ji}$ $\forall i, j$

• $\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$

pf: ① $X_{ij} = \partial_i \partial_j X = \partial_j \partial_i X = X_{ji}$

$\Rightarrow \Gamma^k_{ij} X_k + h_{ij} N = \Gamma^k_{ji} X_k + h_{ji} N$

$\Rightarrow \begin{cases} \Gamma^k_{ij} = \Gamma^k_{ji} \\ h_{ij} = h_{ji} \end{cases}$ using linear indep.

#

Einstein summation convention: repeated indices means summation.

$$\partial_j g_{il} = \partial_j \langle X_i, X_l \rangle$$
$$= \langle X_{ij}, X_l \rangle + \langle X_i, X_{jl} \rangle$$
$$= \Gamma_{ij}^k g_{kl} + \Gamma_{jl}^k g_{ik}.$$

$$\Rightarrow \begin{cases} (+)\ \partial_j g_{il} = \Gamma_{ij}^k \underline{g_{kl}} + \Gamma_{jl}^k g_{ik} \\ \qquad\qquad\quad \shortparallel \\ (+)\ \partial_i g_{jl} = \Gamma_{ij}^k \underline{g_{kl}} + \Gamma_{il}^k g_{jk} \\ (-)\ \partial_l g_{ij} = \Gamma_{il}^k g_{kj} + \Gamma_{jl}^k g_{ik} \end{cases}$$

$$\Rightarrow \left(2\Gamma_{ij}^k g_{kl} = \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right) \cdot g^{lp}$$

where $\Gamma_{ij}^k g_{kl} \cdot g^{lp} = \Gamma_{ij}^k \delta_k^p = \Gamma_{ij}^p.$

$\Rightarrow$ Result (by replacing k with p) #

& $\mathrm{II}_{ij} = \langle N, X_{ij} \rangle = h_{ij}$

& $N_i = a_i^j X_j \quad (\because N_i \perp N)$

$$\Rightarrow -\langle N, X_{ik} \rangle = \langle N_i, X_k \rangle = a_i^j g_{jk}$$

$$\Rightarrow a_i^j g_{jk} = -\mathrm{II}_{ik}$$

$$\Rightarrow a_i^j = -\mathrm{II}_{ik} g^{kj} \nearrow \quad N_i = -\mathrm{II}_{ik} g^{kj} X_j$$

higher order : $X_{ijk} = X_{kij}$

$$X_{ijm} = \left(\mathrm{II}_{ij} N + \Gamma_{ij}^{k} X_k \right)_m$$

$$= \partial_m \mathrm{II}_{ij} \cdot N - \mathrm{II}_{ij} \mathrm{II}_{mp} g^{pk} X_k$$

$$+ \partial_m \Gamma_{ij}^{k} \cdot X_k + \Gamma_{ij}^{k} \cdot \left(\Gamma_{mk}^{p} X_p + \mathrm{II}_{km} N \right)$$

$$= \left(\partial_m \mathrm{II}_{ij} + \Gamma_{ij}^{k} \mathrm{II}_{km} \right) N$$

$$+ \left(\partial_m \Gamma_{ij}^{k} - \mathrm{II}_{ij} \mathrm{II}_{mp} g^{pk} + \Gamma_{ij}^{p} \Gamma_{mp}^{k} \right) X_k.$$

Since $X_{ijm} = X_{imj}$,

$$\Rightarrow \begin{cases} ① \ \partial_m \mathrm{II}_{ij} + \Gamma_{ij}^{k} \mathrm{II}_{km} = \partial_j \mathrm{II}_{im} + \Gamma_{im}^{k} \mathrm{II}_{kj} \\ ② \ \partial_m \Gamma_{ij}^{k} - \mathrm{II}_{ij} \mathrm{II}_{mp} g^{pk} + \Gamma_{ij}^{p} \Gamma_{mp}^{k} \\ \quad = \partial_j \Gamma_{im}^{k} - \mathrm{II}_{im} \mathrm{II}_{jp} g^{pk} + \Gamma_{im}^{p} \Gamma_{jp}^{k}, \quad \forall i, j, m, k. \end{cases}$$

(Revisit later)

② with $m = k$, and sum over $k = 1, 2$, $\Rightarrow$

$$\partial_k \Gamma_{ij}^{k} - \partial_j \Gamma_{ik}^{k} + \Gamma_{ij}^{p} \Gamma_{kp}^{k} - \Gamma_{ik}^{p} \Gamma_{jp}^{k}$$

$$= \left(\mathrm{II}_{ij} \mathrm{II}_{kp} - \mathrm{II}_{ik} \mathrm{II}_{jp} \right) g^{pk}.$$

$$= \mathrm{II}_{ij} \mathrm{II}_{kp} g^{pk} - \mathrm{II}_{ik} \mathrm{II}_{jp} g^{pk}, \quad \forall i, j$$

$$\Rightarrow g^{ij} \left(\partial_k \Gamma_{ij}^{k} - \partial_j \Gamma_{ik}^{k} + \Gamma_{ij}^{p} \Gamma_{kp}^{k} - \Gamma_{ik}^{p} \Gamma_{jp}^{k} \right)$$

$$= (g^{ij} \mathrm{II}_{ij})(g^{pk} \mathrm{II}_{pk}) - g^{ij} \mathrm{II}_{ik} \mathrm{II}_{jp} g^{pk}$$

Recall : the 2nd f.f. $\mathrm{II}(v,w) \triangleq \langle S_p(v), w \rangle$

$$= g(S_p(v), w)$$

$$\Rightarrow \quad \mathrm{II}_{ij} = S_i^k g_{kj} \quad , \text{ if } \quad S_p(X_i) = S_i^j X_j .$$

$$\therefore \; = (\operatorname{tr} S)^2 - S_k^j \cdot S_j^k$$

$\boxed{2D}$

$$= (S_1^1 + S_2^2)^2 - (S_1^1 S_1^1 + S_1^2 S_2^1 + S_2^1 S_1^2 + S_2^2 S_2^2)$$

$$= 2(S_1^1 S_2^2 - S_1^2 S_2^1)$$

$$= 2 \det[S_p] = 2K(p) \quad \#$$

$$\therefore K = \frac{1}{2}\left(g^{ij}\left(\partial_k \Gamma_{ij}^k - \partial_j \Gamma_{ik}^k + \Gamma_{ij}^p \Gamma_{kp}^k - \Gamma_{ik}^p \Gamma_{jp}^k\right)\right)$$

(In higher dimensions, it is called the scalar curv.)

All terms only depends on Γ_{ij}^k and its variations

when Γ_{ij}^k depends only on g_{ij} !!

$\Rightarrow$ intrinsic

$\#$

From: ① $\partial_m \mathrm{II}_{ij} - \Gamma_{im}^p \mathrm{II}_{pj} = \partial_i \mathrm{II}_{jm} - \Gamma_{ij}^p \mathrm{II}_{mp}$

is called the Codazzi identity.

Moreover, since $N_{ij} = N_{ji} \quad \forall i,j$

and $N_{ij} = \partial_j \left(- \mathrm{II}_{im} \, g^{mp} \, X_p \right)$

$$= - \partial_j \left(S_i^p X_p \right)$$

$$= - \partial_j S_i^p \cdot X_p - S_i^p \left(\Gamma_{jp}^q X_q + \mathrm{II}_{pj} N \right)$$

$\therefore$

$$\begin{cases} S_i^p \, \mathrm{II}_{pj} = S_j^p \, \mathrm{II}_{pi} \quad \text{(no extra information)} \\ \partial_j S_i^p + S_i^l \Gamma_{jl}^p = \partial_i S_j^p + S_j^l \Gamma_{il}^p , \quad \forall i,j \end{cases}$$

$\star$

$\#$