Math 4030, HW 4. Due: 13 Nov 2023

(1) Given a regular surface $S \subset \mathbb{R}^3$ with orientation N. Let $p \in S$, show that the mean curvature H at $p \in S$ is given by

$$H = \frac{1}{\pi} \int_0^{\pi} k_n(\theta)\, d\theta$$

where $k_n(\theta)$ is the normal curvature at p along the direction making an angle θ with a fixed direction.

(2) Show that the sum of the normal curvatures for any pair of orthogonal directions at a point $p \in S$ is a constant.

(3) Find the Gaussian curvature and mean curvature of the surface $z = axy$ where $a \neq 0$.

(4) Let S be a regular surface with orientation N. Let $V \subset S$ be an open set in S and $f : V \to \mathbb{R}$ be any nowhere-zero differentiable function on V. Let v_1, v_2 be two differentiable vector on V such that at each $p \in V$, $\{v_1, v_2\}$ are orthonormal and $N = v_1 \times v_2$.

(a) Show that the Gaussian curvature K of V is given by

$$K = \frac{1}{f^3} \langle fN, d(fN)(v_1) \times d(fN)(v_2) \rangle.$$

(b) Using (a), show that if f is the function

$$\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$$

restricted on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

then the Gaussian curvature is given by $K = (abc)^{-2} f^{-4}$.