Math 4030, HW 3. Due: 30 Oct 2023

(1) Compute the first fundamental form of the following parametrized surfaces where they are regular:
 (a) $X(u,v) = (a\sin u\cos v, b\sin u\sin v, c\cos u)$;
 (b) $X(u,v) = (au\cosh v, bu\sinh v, u^2)$.

(2) Show that

$$X(u,v) = (u\sin\alpha\cos v, u\sin\alpha\sin v, u\cos\alpha)$$

where $u \in (0,+\infty, v \in (0,2\pi)$ and α is a constant, is a parametrization of the cone with the angle of vertex 2α. Prove that the curve given by

$$\gamma(t) = X(c\exp(t\cdot\sin\alpha\cot\beta), t)$$

where c,β are constants, intersects the generators of the cone (i.e. $v = \text{const.}$) under the constant angle β.

(3) The gradient of a differentiable function $f : S \to \mathbb{R}$ on a regular surface S is a differentiable map $\text{grad}(f) : S \to \mathbb{R}^3$ which assigns to each point $p \in S$ a vector $\text{grad}(f)_p \in T_pS$ so that for all $v \in T_pS$,

$$\langle \text{grad}(f)_p, v\rangle = df_p(v).$$

Here we are regarding df_p as a real-valued linear function defined on T_pS.
 (a) Express $\text{grad}(f)$ in term of the coefficient of the first fundamental form (i.e. E, F, G), X_u, X_v and the partial derivatives of f on the local parametrization $X : U \to S$ of S at $p \in X(U)$.
 (b) Let $p \in S$ and $\text{grad}(f)_p \neq 0$. Show that $v \in T_pS$ with $|v| = 1$ satisfies

$$df_p(v) = \max\{df_p(u) : u \in T_pS, |u| = 1\}$$

 if and only if $v = \text{grad}(f)_p/|\text{grad}(f)_p|$.

(4) Suppose S is a regular surface with orientation N so that $dN_p \neq 0$ for all $p \in S$. If the mean curvature H vanishes on S, show that the Gauss map $N : S \to \mathbb{S}^2$ satisfies

$$\langle dN_p(v), dN_p(w)\rangle = -K_p\langle v, w\rangle$$

for all $p \in S$ and $v, w \in T_pS$. Here K_p denotes the Gaussian curvature at $p \in S$.