Math 2050, HW 1. Due: 2 Oct 2023

(1) Given the parametrized curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ where

$$\alpha(s) = \left(a \cos \frac{s}{c}, a \sin \frac{s}{c}, b\frac{s}{c}\right)$$

and $c^2 = a^2 + b^2$.

  (a) Show that the parameter s is the arc-length;
  (b) Find the curvature and torsion of α;
  (c) Show that the tangent lines to α make a constant angle with the z-axis.

(2) Show that a regular curve is a circular helix if the curvature $\kappa > 0$ is a constant and the torsion $\tau \neq 0$ is a constant.

(3) Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a regular curve such that $||\gamma(t)|| \leq 1$ for all $t \in \mathbb{R}$. If there is $t_0 \in \mathbb{R}$ so that $||\gamma(t_0)|| = 1$, show that $|\kappa(t_0)| \geq 1$.

(4) Suppose a planar curve is given by $\rho = \rho(\theta), a \leq \theta \leq b$ in the polar coordinate.

  (a) Show that the arc-length is given by

$$\int_a^b \sqrt{\rho^2 + (\rho')^2}\, d\theta.$$

  (b) Express the curvature $\kappa(\theta)$ of the curve in term of ρ.

(5) Suppose all normal of a parametrized curve $\alpha : I \to \mathbb{R}^3$ pass through a fixed point, show that the trace $\alpha(I)$ of the curve is contained in a circle.