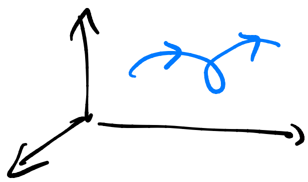
Differential Geom. Week 1:

Goal of 4030: Study Geometric object in $\mathbb{R}^3$.

Curves | Surfaces



---

differentiable curves:

Defn: A parametrized differentiable curve in $\mathbb{R}^3$ is a differentiable map $\alpha : I = (a,b) \longrightarrow \mathbb{R}^3$ (OR $(-\infty, \infty)$)

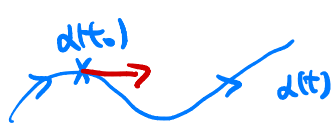
i.e. $\alpha : t \longmapsto (x^1(t), x^2(t), x^3(t)) \in \mathbb{R}^3$. where

$x^i(\cdot)$ are differentiable functions

$t :=$ parameter of curve.

Rmk: Think of $\alpha(t)$ as position vector,

t as time parameter.



$\rightarrow = \alpha'(t_0)$

Here $\alpha'(t)$ is called the tangent vector of curve α at t

where $\alpha'(t) = \left(\frac{\partial}{\partial t} x^1, \frac{\partial}{\partial t} x^2, \frac{\partial}{\partial t} x^3\right) \in \mathbb{R}^3$ (= tangent space of $\mathbb{R}^3$)

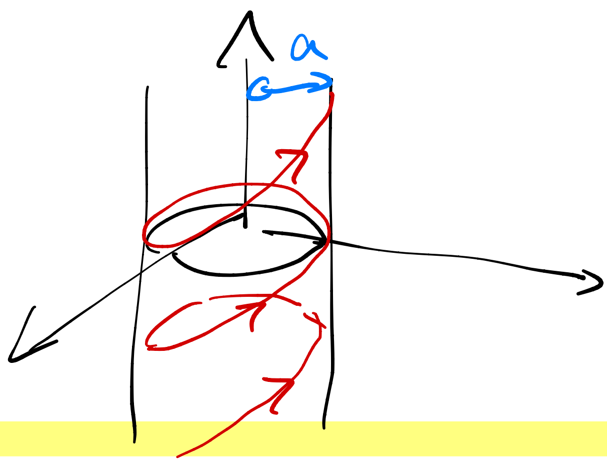
Defn: $\alpha(I) \subseteq \mathbb{R}^3$ is called the trace of α.

(good)

eg: Parametrized differentiable curves :

$$\alpha(t) = (a\cos t,\ a\sin t,\ bt), \quad t \in \mathbb{R}.$$

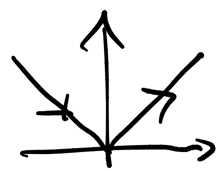
where $a, b > 0$ are fixed real no.



BAD Example : $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by

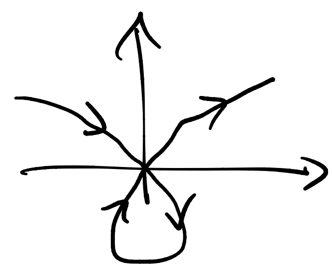
$$\alpha(t) = (t, |t|)$$

is Not differentiable. (at $t=0$)



Example : $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by

$$\alpha(t) = (t^3 - 4t,\ t^2 - 4)$$

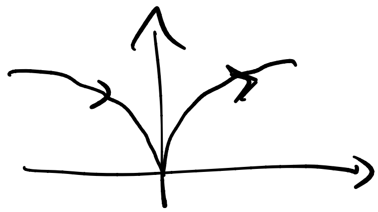


$$\alpha(2) = \alpha(-2) = (0,0) \in \mathbb{R}^2$$

α is a parametrized differentiable curve but Not one-one.

Example : $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\alpha(t) = (t^3, t^2)$$



$\alpha'(t) = (3t^2, 2t)$

and

$\alpha'(0) = 0$. $\leftarrow$ velocity $= 0$ at $t = 0$.

<u>Example:</u> $\alpha, \beta : \mathbb{R} \to \mathbb{R}^2$ given by

$\alpha(t) = (\cos t, \sin t)$

$\beta(t) = (\sin t, -\cos t)$

$\alpha(\mathbb{R}) = \beta(\mathbb{R}) = \{x^2+y^2=1\}$

and $\alpha(t) = \beta(\frac{\pi}{2}+t)$, $t \in \mathbb{R}$.

$\rightsquigarrow$ α is a reparametrization of β.

More generally, let $\alpha : (a,b) \to \mathbb{R}^3$ be a parametrized curve, $f : (c,d) \to (a,b)$ with $t = f(s)$, $f' > 0$ then $\beta(s) = \alpha(f(s))$ is a reparametrization.

* . We are (mostly) interested in curve $\alpha : I \to \mathbb{R}^3$ which is regular (i.e. $\alpha' \neq 0$ $\forall t \in I$).

Call such curve α to be regular curve.

given $\alpha : I \to \mathbb{R}^3$

$\alpha(t) = (x(t), y(t), z(t))$ = position.

- $\alpha'(t) = (x'(t), y'(t), z'(t))$ = velocity vector.
- $|\alpha'(t)|$ = norm of velocity = speed.

$$= \sqrt{\langle \alpha', \alpha' \rangle_{\mathbb{R}^3}}$$

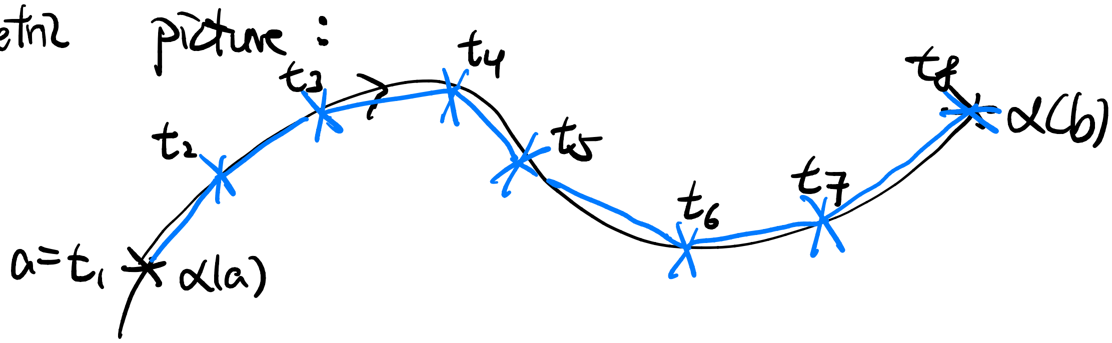
$$= \sqrt{(x')^2 + (y')^2 + (z')^2}$$

(= differentiable if α = regular.)

Defn: given $[a,b] \subseteq I$, the arc-length of curve $\alpha|_{[a,b]}$ is given by

$$L(\alpha|_{[a,b]}) = \int_a^b |\alpha'(t)|\, dt.$$

Geometric picture:



length of $\alpha|_{[a,b]} = L(\alpha|_{[a,b]})$

$$\approx \sum_{i=1}^{7} |\alpha(t_{i+1}) - \alpha(t_i)|.$$

$$\approx \sum_{i=1}^{7} \int_{t_i}^{t_{i+1}} |\alpha'|\, ds \approx \int_a^b |\alpha'|\, ds.$$

Exercise : make it rigorous

$\forall \varepsilon>0, \exists \delta>0$ s.t. if $|P|<\delta$,

then $\left| \int_a^b |\alpha'| \, dt - \sum_{i=1}^{l} |\alpha(t_{i+1}) - \alpha(t_i)| \right| < \varepsilon.$

Given a regular curve $\alpha : I=(a,b) \to \mathbb{R}^3$.

Fix $t_0 \in I$, for $t \in (t_0, b)$

define $s(t) = \int_{t_0}^{t} |\alpha'| \, ds$

$= L(\alpha|_{[t_0,t]})$.

* α is parametrized by arc-length iff $|\alpha'(t)| \equiv 1$.

(such that t refers to arc-length from fixed t_0.)

* Define $\tilde{\alpha}(s) = \alpha(t(s))$.

then $\tilde{\alpha}$ is parametrized by arc-length if

$$|\tilde{\alpha}'(s)| = |\alpha'(t)| \cdot \frac{dt}{ds} = 1$$

$\therefore$ Require : $t(s)$ satisfies $\frac{dt}{ds} = \frac{1}{|\alpha'(t)|} > 0$.

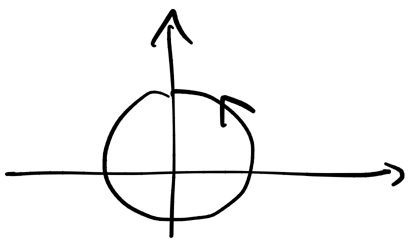
$$\Rightarrow \frac{ds}{dt} = |\alpha'(t)| > 0.$$

Inverse fcn thm $\Rightarrow$ $s(t)$ has a differentiable inverse $t(s)$, s.t.

$$\frac{dt}{ds} = \frac{1}{|\alpha'(t(s))|}$$

$\Rightarrow$ $\alpha(s)$ exists and is an arc-length parametrization. $\leftarrow$ why care??

eg:



$\alpha(t) = (\cos t, \sin t)$, $t \in [0, 2\pi)$

$\alpha'(t) = (-\sin t, \cos t)$

where $|\alpha'| \equiv 1$.

But we can also parametrize it as

$\beta(t) = (\cos t^2, \sin t^2)$ (OR something weird)

Same geometry but different speed.

only focus on α. (if possible) $\leftarrow$ who care ☺

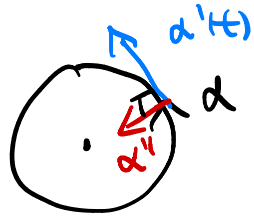
Given curve with arc-length parametrization.

$\alpha : I \to \mathbb{R}^3$, $|\alpha'(t)| \equiv 1$.

$|\alpha'| \equiv 1 \Rightarrow \langle \alpha', \alpha' \rangle \equiv 1$ (= constant)

$$\Rightarrow \quad \langle \alpha', \alpha'' \rangle = 0 \qquad \text{i.e.} \quad \alpha'' \perp \alpha'.$$

eg:



Rotation with uniform speed.

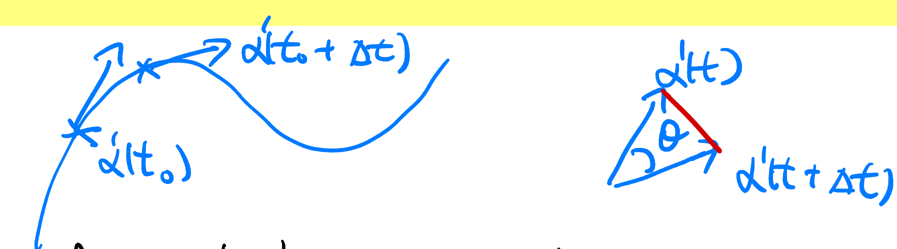
$\alpha(t) = (\cos t, \sin t), \quad t \in \mathbb{R}.$

- $\alpha''(t)$ = acceleration (= centripetal force)

$$= (-\cos t, -\sin t)$$

- $\alpha''(t) \perp \alpha'(t)$ (unit speed $\Rightarrow$ acceleration = change of direction)

Geometrically,



$$|\alpha''(t_0)| = \lim_{\Delta t \to 0} \left| \frac{\alpha'(t_0+\Delta t) - \alpha'(t_0)}{\Delta t} \right|$$

$$= \lim_{\Delta t \to 0} \frac{|\alpha'(t)|}{|\Delta t|} \left| \sin\left(\frac{\Delta\theta}{2}\right) \cdot 2 \right|$$

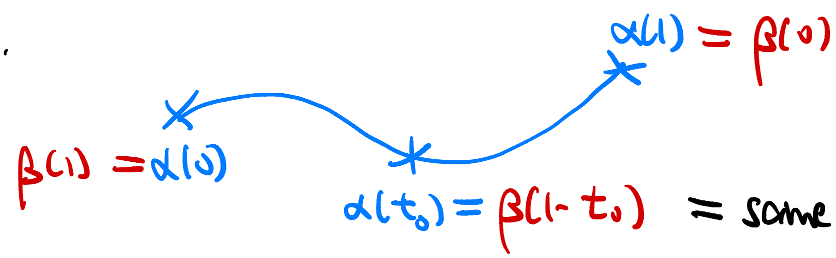
$$= \lim_{\Delta t \to 0} \left| \frac{\Delta \theta}{\Delta t} \right| .$$

Defn: Given a arc-length parametrized curve

$\alpha: I \to \mathbb{R}^3$. The no. $|\alpha''(s)| = k(s)$ is called the curvature of α at s.

Rmk: Curvature is independent of orientation.

i.e.



$= $ same point

$\alpha(t) = \beta(1-t)$, $t \in [0,1]$.

α, β are in opposite direction. although also in arc-length parametrization

$$(|\alpha'(t)| = |\beta'(1-t)| = 1)$$

$$|\alpha''(t)| = \text{curvature at } \underset{\beta(1-t)}{\underset{\shortparallel}{\alpha(t)}} = |\beta''(1-t)|$$

More than that $\alpha''(t) = \beta''(1-t)$!!

eg:

At point s.t. $k \neq 0$, define a unit vector

$$N(s) = \frac{\alpha''(s)}{|\alpha''(s)|} \perp \alpha'(s).$$

Call it normal vector at s.

$\Sigma \triangleq \operatorname{span}\{N(s), \underset{\alpha'(s)}{\underset{\shortparallel}{T(s)}}\}$ is called osculating plane at s.

Defn: The unit vector $B(s) \triangleq T(s) \times N(s)$ (if $k \neq 0$) is called binormal vector at s where

$$\begin{cases} B(s) \perp T(s), N(s) \\ |B(s)| \equiv 1 \end{cases}$$

At $\alpha(s)$ with $k(s) \neq 0$, we have o.N. frame

$$\{T(s), N(s), B(s)\}.$$

Calculus $\Rightarrow$
$$\begin{cases} T' = k \cdot N & \text{(defn of } N(s)\text{)}. \\ N' = aT + bB \\ B' = cT + dN. \end{cases}$$

$$\begin{aligned} c = \langle B', T\rangle &= \langle (T \times N)', T\rangle \\ &= \langle \cancelto{0}{T' \times N} + T \times N', T\rangle \\ &= \langle T \times N', T\rangle = 0. \end{aligned}$$

$$\begin{aligned} d = \langle B', N\rangle &= \langle T \times N', N\rangle \\ &= b\langle T \times B, N\rangle = -b. \end{aligned}$$

$$a = \langle N', T\rangle = -\langle N, T'\rangle = -k.$$

$$\therefore \begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where k = curvature at s

and $\tau \triangleq$ torsion (of α) at s.

Q: OK, but why care ??

<u>Prop</u>: Suppose $k(s) \equiv 0$ for an arc-length parametrized curve, then $\alpha(s)$ = straight line.

(And straight $\Rightarrow k(s) \equiv 0$)

pf: $\alpha''(s) = 0 = (x''(t), y''(t), z''(t))$

$\Rightarrow \quad x'(t), \ y'(t), \ z'(t) \equiv$ constant

$$\Rightarrow \begin{cases} x(t) = x_0 + at \\ y(t) = y_0 + bt \\ z(t) = z_0 + ct \end{cases} \quad \text{where} \quad a^2 + b^2 + c^2 = 1.$$

$\Rightarrow \quad \alpha(t)$ = straight line #

<u>prop</u>: Suppose $k > 0$, then $\tau \equiv 0$ iff α is a plane curve.

pf: ($\Leftarrow$) if α = plane curve

then $\{T(s), N(s)\} \in \Sigma$, $\forall s \in I$

for some 2-plane Σ.

$$\Rightarrow \quad T \times N = B(s) \equiv \text{Normal of } \Sigma.$$

$$\Rightarrow \quad B'(s) = -\tau B(s) \equiv 0.$$

$$\Rightarrow \quad \tau \equiv 0.$$

($\Rightarrow$): if $\tau \equiv 0$, then $B(s) \equiv B_0$ $\forall s$.

$$\Rightarrow \quad \langle \alpha(s) - \alpha(s_0), B_0 \rangle'$$

$$= \langle \alpha'(s), B_0 \rangle = 0$$

$$\Rightarrow \quad \langle \alpha(s) - \alpha(s_0), B_0 \rangle \equiv 0 \quad \text{on } I.$$

$$\Rightarrow \quad \alpha = \text{plane curve}.$$

---

eg (circular motion): $\alpha(t) = (r\cos t, r\sin t)$

From high school physics, $a = \frac{v^2}{r}$

Now $v = 1$ (arc-length parametrization)

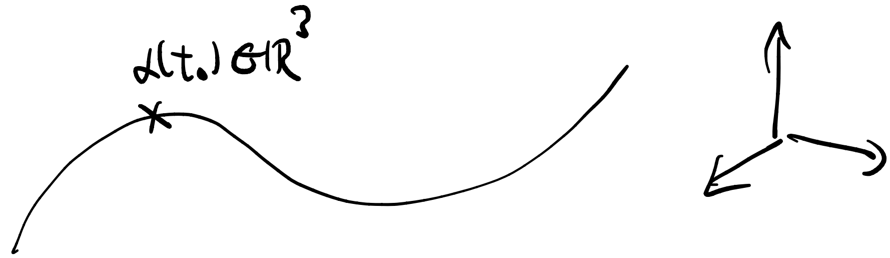
$a = \frac{1}{r}$ = curvature $\rightsquigarrow$ Generally, $r \triangleq \frac{1}{\kappa}$

is called radius of curv..

Given a regular curve $\alpha: I \to \mathbb{R}^3$ which is not parametrized by arc-length,

Q: How to compute $\kappa(t)$ and $\tau(t)$??

at



$$\alpha' = |\alpha'|\, T \qquad \text{where } |\alpha'| \neq 0.$$

$$\Rightarrow \quad \alpha'' = |\alpha'|'\, T + |\alpha'|\, T'$$

differentiable

$$T' = \frac{dT}{ds} \cdot \frac{ds}{dt} = \kappa N \cdot |\alpha'|.$$

$$\left(\because\ s(t) = \int_a^t |\alpha'|\, d\tau \right)$$

$$\Rightarrow \quad \alpha'' \times T = |\alpha'|\, T' \times T$$

$$= \kappa |\alpha'|^2\, N \times T$$

$$\Rightarrow \quad \kappa = \frac{|\alpha'' \times T|}{|\alpha'|^2} = \frac{|\alpha'' \times \alpha'|}{|\alpha'|^3} \quad \#$$

Torsion:

$$\alpha''' = |\alpha'|'' T + |\alpha'|' T' + \left(k|\alpha'|^2 N\right)'$$

$$= f(t)\, T + g(t)\, N + k|\alpha'|^2 N'$$

$$= f\, T + \tilde{g} N + k|\alpha'|^3 \tau B$$

$$\Rightarrow \quad \tau = \frac{\langle \alpha''', T \times N \rangle}{|\alpha'|^3 k}.$$

$$= \frac{\langle \alpha''', \boxed{T \times N} \rangle}{|\alpha' \times \alpha''|}$$

find this !!

Recall, $\begin{cases} \alpha' = |\alpha'| T \\ \alpha'' = |\alpha'|' T + k|\alpha'|^2 N \end{cases}$

$$\Rightarrow \alpha' \times \alpha'' = |\alpha'| T \times k|\alpha'|^2 N$$

$$\Rightarrow \quad T \times N = \frac{\alpha' \times \alpha''}{k|\alpha'|^3}.$$

$$= \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|}$$

$$\Rightarrow \quad \tau = \frac{\langle \alpha''', \alpha' \times \alpha'' \rangle}{|\alpha' \times \alpha''|^2} \quad \#$$