19/10/23

MATH4030 Tutorial

Announcements:

- HW3 due 30/10

Review Midterm Problems:

1) a) $\alpha(t) = (\cos(5t), \sin(5t), 12t)$.

Find the speed and arc-length param. β of α.

$\alpha'(t) = (-5\sin(5t), 5\cos(5t), 12)$

$|\alpha'(t)| = 13$.

$s(t) = \int_0^t |\alpha'(u)|du = \int_0^t 13du = 13t$. $\Rightarrow \beta(s) = \left(\cos\left(\frac{5}{13}s\right), \sin\left(\frac{5}{13}s\right), \frac{12}{13}s\right)$ //.

b) Compute K:

More complicated: $K(t) = \dfrac{|\alpha' \times \alpha''|}{|\alpha'|^3}$.

Simpler: use $\beta(s)$. $K(s) = |\beta''(s)| = \left(\frac{5}{13}\right)^2$.

c) Compute the Frenet frame $\{T, N, B\}$ for β.

$T(s) = \beta'(s)$. $N = \dfrac{\beta''(s)}{|\beta''(s)|}$. $B = T \times N$.

d) Compute τ: $B' = -\tau N$.

2) A helix is a regular curve s.t. $\exists$ fixed direction u, s.t. tangent to the curve meets u at a constant angle. Show that if α is a helix, then $\frac{\kappa}{\tau}$ is constant. $(\kappa > 0, \tau \neq 0)$.

Pf: helix condition means: $\langle T, u\rangle = \cos\theta_0 = \text{const}$.

WLOG we take u to be a unit vector.

differentiating both sides, we get $0 = (\langle T, u\rangle)' = \langle T', u\rangle + \langle T, \cancel{u'}\rangle$

$\langle T, u'\rangle = 0$

WLOG take α param. by arclength.

$= \langle \kappa N, u\rangle$.

$\kappa > 0 \Rightarrow \langle N, u \rangle = 0 \Rightarrow u$ is perpendicular to normal vector N.

So writing u in the $\{T, N, B\}$ basis, we have

$$u = \cos\theta_0 T + 0 \cdot N + \underbrace{\sin\theta_0 B}_{yB}. \quad \leftarrow \text{b/c. } u \text{ is a unit vector then } 1 = \sqrt{\cos^2\theta_0 + y^2}.$$

$$u = \cos\theta_0 T + \sin\theta_0 B.$$

Differentiating again gives

$$0 = \kappa\cos\theta_0 N - \tau\sin\theta_0 N. \Rightarrow \kappa\cos\theta_0 = \tau\sin\theta_0.$$

$$\Rightarrow \frac{\kappa}{\tau} = \tan\theta_0 = \text{const.}$$

3) a) Compute N s.t. $\langle N, (0,0,1)\rangle < 0$.

$$X_u = (v, 2u, 3v-2u) \qquad X = (uv, u^2-v^2, 3vu-u^2+v^2)$$

$$X_v = (u, -2v, 3u+2v).$$

$$X_u \times X_v = (6u^2+6v^2, -2u^2-2v^2, -2u^2-2v^2).$$

$$|X_u \times X_v| = \sqrt{44}(u^2+v^2) = 2\sqrt{11}(u^2+v^2).$$

$$N = \left(\frac{3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}\right).$$ and clearly $\langle N, (0,0,1)\rangle < 0$.

b) Compute the shape operator for this surface.

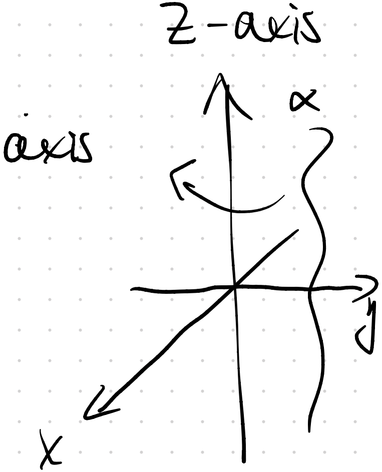
Let $v \in T_pM$. Then let $\alpha: I \to S$ be a regular curve s.t.

$\alpha(0) = p$, $\alpha'(0) = v$.

Then $S_p(v) = -\frac{d}{dt}\Big|_{t=0} N(\alpha(t)) = -\frac{d}{dt}\Big|_{t=0}\left(\frac{3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}\right) = 0.$

4) For a surface of revolution, show that it can be param. s.t.

$E = E(v), \quad F = 0, \quad G = 1.$



Pf: WLOG 1) we can assume the axis of rotation is the z-axis (by rigid motion).

2) Let $\alpha(t)$ be the curve we are revolving. Then we can take t to be arc-length param.

$$\alpha(t) = (f(t), g(t)).$$

by arc-length param, $1 = |\alpha'(t)| = (f'(t))^2 + (g'(t))^2$.

Then we choose the param. for S:

$$X(t,\theta) = (f(t)\cos\theta,\ f(t)\sin\theta,\ g(t)). \qquad t \in I,\ \theta \in [0, 2\pi).$$

Compute 1st ff.

$$X_t = (f'(t)\cos\theta,\ f'(t)\sin\theta,\ g'(t)).$$

$$X_\theta = (-f(t)\sin\theta,\ f(t)\cos\theta,\ 0).$$

$$E = \langle X_\theta, X_\theta\rangle = f(t)^2\sin^2\theta + f(t)^2\cos^2\theta = f(t)^2$$ $v := t$. here.

$$= E(v).$$

$$F = \langle X_t, X_\theta\rangle = 0.$$

$$G = \langle X_t, X_t\rangle = (f'(t))^2 + (g'(t))^2 = 1.$$ by arc-length param.

5) a) Show that for an orientable surface S, $S_p(v) \equiv 0 \Rightarrow S$ is contained in a plane.

Pf: S is orientable, so $\exists$ an orientation/normal vector field N.

$$S_p(v) \equiv 0 \Rightarrow 0 = -\frac{d}{dt}\Big|_{t=0} N(\alpha(t)).$$

$$\Rightarrow dN_p(v) \equiv 0.$$

$\Rightarrow N$ is constant $\Rightarrow S$ is contained in a plane. ✓

b) Show that if $S_p(v)$ satisfies $S_p(v) = k(p)\, v$ for all v for some differentiable function $k: S \to \mathbb{R}$, then $k = k_0$ is constant.

Common pitfalls: This is not the eigenvector equation:

eigenvector eqn. $\exists\, v_0$ s.t. $S_p(v_0) = \lambda_0 v_0$ for some $\lambda_0 \in \mathbb{R}$

But cond. above is for all vectors v.

$S_p(v) = k(p)v \Rightarrow$ if α is a curve w/ $\alpha(0)=p$, $\alpha'(0)=v$, then

$$k(p)v = -\frac{d}{dt}\Big|_{t=0} N(\alpha(t)) = -N'(\alpha(0))\,\alpha'(0)$$
$$= -N'(\alpha(0))\,v.$$

$$\Rightarrow N'(p) = k(p).$$

In particular, using $S_p(X_u) = -N_u$, $S_p(X_v) = -N_v$, we get

$$\begin{cases} k(p)\,X_u = -N_u \\ k(p)\,X_v = -N_v. \end{cases}$$

Recall: $k: S \to \mathbb{R}$, so dk has components k_u, k_v.

Differentiating both eqn's in u and v above,

$$k_u X_u + k X_{uu} = -N_{uu}$$
$$k_u X_v + k X_{vu} = -N_{vu} \quad (1)$$
$$k_v X_u + k X_{uv} = -N_{uv} \quad (2)$$
$$k_v X_v + k X_{vv} = -N_{vv}.$$

$\Rightarrow k_u X_v + \cancel{k X_{vu}} = k_v X_u + \cancel{k X_{uv}}$

$\Rightarrow k_u X_v = k_v X_u.$

$\Rightarrow$ by lin. indep. of X_u, X_v, $k_u = k_v = 0$.

$\Rightarrow dk = 0 \Rightarrow k$ is constant.

One other common pitfall for this question is to confuse the orientation of a surface N, with the normal vector N associated with a curve α.