

Math 3360: Mathematical Imaging

Midterm practice

1. Recall that an image transformation $\mathcal{O} : M_{n \times n}(\mathbb{R}) \rightarrow M_{n \times n}(\mathbb{R})$ is said to be separable if there exist matrices $A \in M_{n \times n}(\mathbb{R})$ and $B \in M_{n \times m}(\mathbb{R})$ such that $\mathcal{O}(f) = AfB$ for any $f \in M_{n \times n}(\mathbb{R})$.

Here $\mathcal{O} : M_{3 \times 3}(\mathbb{R}) \rightarrow M_{3 \times 3}(\mathbb{R})$ is an image transformation and the transformation matrix of its PSF is

$$H = \begin{pmatrix} 2 & 4 & 6 & 2 & 4 & 6 & 0 & 0 & 0 \\ 8 & 10 & 0 & 8 & 10 & 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 & 3 & 6 & 9 \\ 4 & 5 & 0 & 0 & 0 & 0 & 12 & 15 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 18 & 0 & 0 \\ 0 & 0 & 0 & 4 & 8 & 12 & 1 & 2 & 3 \\ 0 & 0 & 0 & 16 & 20 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 24 & 0 & 0 & 6 & 0 & 0 \end{pmatrix}.$$

Please determine if \mathcal{O} is separable. If yes, please find the corresponding matrices $A \in M_{3 \times 3}(\mathbb{R})$ and $B \in M_{3 \times 3}(\mathbb{R})$.

Solution: Let $A = (a_{ij})_{1 \leq i, j \leq 3}$, $B = (b_{ij})_{1 \leq i, j \leq 3}$ and $g = \mathcal{O}(f) \in M_{3 \times 3}(\mathbb{R})$, then we have

$$g_{\alpha, \beta} = \sum_{x=1}^3 a_{\alpha x} \left(\sum_{y=1}^3 f(x, y) b_{y\beta} \right) = \sum_{x=1}^3 \sum_{y=1}^3 a_{\alpha x} b_{y\beta} f(x, y),$$

Which means $h^{\alpha, \beta}(x, y) = a_{\alpha x} b_{y\beta}$. Hence the transformation matrix

$$H = \begin{pmatrix} a_{11}b_{11} & a_{12}b_{11} & a_{13}b_{11} & a_{11}b_{21} & a_{12}b_{21} & a_{13}b_{21} & a_{11}b_{31} & a_{12}b_{31} & a_{13}b_{31} \\ a_{21}b_{11} & a_{22}b_{11} & a_{23}b_{11} & a_{21}b_{21} & a_{22}b_{21} & a_{23}b_{21} & a_{21}b_{31} & a_{22}b_{31} & a_{23}b_{31} \\ a_{31}b_{11} & a_{32}b_{11} & a_{33}b_{11} & a_{31}b_{21} & a_{32}b_{21} & a_{33}b_{21} & a_{31}b_{31} & a_{32}b_{31} & a_{33}b_{31} \\ a_{11}b_{12} & a_{12}b_{12} & a_{13}b_{12} & a_{11}b_{22} & a_{12}b_{22} & a_{13}b_{22} & a_{11}b_{32} & a_{12}b_{32} & a_{13}b_{32} \\ a_{21}b_{12} & a_{22}b_{12} & a_{23}b_{12} & a_{21}b_{22} & a_{22}b_{22} & a_{23}b_{22} & a_{21}b_{32} & a_{22}b_{32} & a_{23}b_{32} \\ a_{31}b_{12} & a_{32}b_{12} & a_{33}b_{12} & a_{31}b_{22} & a_{32}b_{22} & a_{33}b_{22} & a_{31}b_{32} & a_{32}b_{32} & a_{33}b_{32} \\ a_{11}b_{13} & a_{12}b_{13} & a_{13}b_{13} & a_{11}b_{23} & a_{12}b_{23} & a_{13}b_{23} & a_{11}b_{33} & a_{12}b_{33} & a_{13}b_{33} \\ a_{21}b_{13} & a_{22}b_{13} & a_{23}b_{13} & a_{21}b_{23} & a_{22}b_{23} & a_{23}b_{23} & a_{21}b_{33} & a_{22}b_{33} & a_{23}b_{33} \\ a_{31}b_{13} & a_{32}b_{13} & a_{33}b_{13} & a_{31}b_{23} & a_{32}b_{23} & a_{33}b_{23} & a_{31}b_{33} & a_{32}b_{33} & a_{33}b_{33} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11}A & b_{21}A & b_{31}A \\ b_{12}A & b_{22}A & b_{32}A \\ b_{13}A & b_{23}A & b_{33}A \end{pmatrix} = B^T \otimes A.$$

At the same time, it's easy to notice that H is the kronecker product of two 3×3 matrices; explicitly, let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 6 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 4 \\ 0 & 3 & 1 \end{pmatrix}$ then $H = B^T \otimes A$. So the image transformation \mathcal{O} is separable and the required A, B are given above.

2. A matrix $H \in M_{n^2 \times n^2}(\mathbb{R})$ is called block-circulant if it has the form

$$H = \begin{pmatrix} H_1 & H_n & \cdots & H_2 \\ H_2 & H_1 & \cdots & H_3 \\ \vdots & \vdots & \ddots & \vdots \\ H_n & H_{n-1} & \cdots & H_1 \end{pmatrix},$$

where $H_i \in M_{n \times n}(\mathbb{R})$ for $i = 1, \dots, n$. Given matrix $k, f \in M_{2 \times 2}(\mathbb{R})$, let the image transformation $\mathcal{O}(f) = k * f$, please prove that the transformation matrix H of \mathcal{O} is block-circulant.

Solution: Let $k = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$ and $g = \mathcal{O}(f) \in M_{2 \times 2}(\mathbb{R})$. Then we have

$$g_{\alpha, \beta} = \sum_{x=1}^2 \sum_{y=1}^2 k_{\alpha-x, \beta-y} f(x, y),$$

which means $h^{\alpha, \beta}(x, y) = k_{\alpha-x, \beta-y}$. Hence the transformation matrix

$$\begin{aligned} H &= \begin{pmatrix} k_{22} & k_{12} & k_{21} & k_{11} \\ k_{12} & k_{22} & k_{11} & k_{21} \\ k_{21} & k_{11} & k_{22} & k_{12} \\ k_{11} & k_{21} & k_{12} & k_{22} \end{pmatrix} \\ &= \begin{pmatrix} H_1 & H_2 \\ H_2 & H_1 \end{pmatrix} \end{aligned}$$

where $H_1 = \begin{pmatrix} k_{22} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}$ and $H_2 = \begin{pmatrix} k_{21} & k_{11} \\ k_{11} & k_{21} \end{pmatrix}$. Hence the transformation matrix H of \mathcal{O} is block-circulant.

3. Let $H = \begin{pmatrix} r & 2r & u & 2u \\ 3r & r & 3v & v \\ 3 & 6 & s & 2s \\ 9 & 3 & 3s & s \end{pmatrix}$ be the transformation matrix corresponding to an image trans-

formation $\mathcal{O} : M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$, where r, s, u, v are all non-zero real numbers. Prove that \mathcal{O} is separable and if and only if $u = v$. Please explain your answer with details.

Solution: \Rightarrow : If $u = v$, we have $H = \begin{pmatrix} r & u \\ 3 & s \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$, then

$$\mathcal{O}(f) = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} f \begin{pmatrix} r & 3 \\ u & s \end{pmatrix}$$

for all $f \in M_{2 \times 2}(\mathbb{R})$. Hence \mathcal{O} is separable.

\Leftarrow : If \mathcal{O} is separable, then there exist $A, B \in M_{2 \times 2}(\mathbb{R})$ such that $\mathcal{O}(f) = AfB$ for all $f \in M_{2 \times 2}(\mathbb{R})$. So the transformation matrix H of \mathcal{O} is given by

$$H = \begin{pmatrix} b_{11}a_{11} & b_{11}a_{12} & b_{21}a_{11} & b_{21}a_{12} \\ b_{11}a_{21} & b_{11}a_{22} & b_{21}a_{21} & b_{21}a_{22} \\ b_{12}a_{11} & b_{12}a_{12} & b_{22}a_{11} & b_{22}a_{12} \\ b_{12}a_{21} & b_{12}a_{22} & b_{22}a_{21} & b_{22}a_{22} \end{pmatrix}$$

where a_{ij} and b_{ij} are the entries of A and B respectively and $1 \leq i, j \leq 2$. Since we have $b_{12}a_{11} = b_{12}a_{22} = 3$, then $a_{11} = a_{22} \neq 0$ and so $u = b_{21}a_{11} = b_{21}a_{22} = v$.

4. Let $H = \begin{pmatrix} 2 & 4 & a & 6 \\ 4 & 2 & 6 & 1 \\ 1 & 6 & 2 & 4 \\ c & 1 & 4 & b \end{pmatrix}$ be the transformation matrix corresponding to an image trans-

formation $\mathcal{O} : M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$, where a, b, c are all non-zero real numbers. Please determine a, b, c such that \mathcal{O} is an image transformation defined by convolution.

Solution: We have proved the transformation matrix H of \mathcal{O} is block-circulant if \mathcal{O} is an image transformation defined by convolution. So $a = 1, b = 2, c = 6$.

5. Let $f = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 4 & 0 & 2 \end{pmatrix}$.

- (a) Compute an SVD of f .
 (b) Express f as a linear combination of its elementary images.

Solution:

(a) $ff^T = \begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix}$.

For $\lambda = 20$:

$$\left[\begin{array}{cc|c} -10 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right],$$

which gives unit eigenvector $\vec{u}_1 = (0, 1)^T$.

For $\lambda = 10$:

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -10 & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right],$$

which gives unit eigenvector $\vec{u}_2 = (1, 0)^T$.

Then $\vec{v}_1 = \frac{f^T \vec{u}_1}{\sigma_1} = \frac{1}{2\sqrt{5}} \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}}(0, 2, 0, 1)^T$, and

$$\vec{v}_2 = \frac{f^T \vec{u}_2}{\sigma_2} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{10}}(1, 0, 3, 0)^T.$$

$$f^T f = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 16 & 0 & 8 \\ 3 & 0 & 9 & 0 \\ 0 & 8 & 0 & 4 \end{pmatrix}.$$

For $f^T f \vec{v} = 0$,

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 16 & 0 & 8 & 0 \\ 3 & 0 & 9 & 0 & 0 \\ 0 & 8 & 0 & 4 & 0 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right],$$

which gives orthonormal eigenvectors $\vec{v}_3 = \frac{1}{\sqrt{10}}(-3, 0, 1, 0)^T$ and $\vec{v}_4 = \frac{1}{\sqrt{5}}(0, 1, 0, -2)^T$.

Hence an SVD of f is $f = U\Sigma V^T$, where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Sigma = \begin{pmatrix} 2\sqrt{5} & 0 & 0 & 0 \\ 0 & \sqrt{10} & 0 & 0 \end{pmatrix} \text{ and } V = \frac{1}{\sqrt{10}} \begin{pmatrix} 0 & 1 & -3 & 0 \\ 2\sqrt{2} & 0 & 0 & \sqrt{2} \\ 0 & 3 & 1 & 0 \\ \sqrt{2} & 0 & 0 & -2\sqrt{2} \end{pmatrix}.$$

(b) The eigenimages are given by

$$\vec{u}_1 \vec{v}_1^T = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0 \ 2 \ 0 \ 1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \end{pmatrix} \text{ and}$$

$$\vec{u}_2 \vec{v}_2^T = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \ 0 \ 3 \ 0) = \begin{pmatrix} \frac{\sqrt{10}}{10} & 0 & \frac{3\sqrt{10}}{10} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\text{Hence } f = 2\sqrt{5} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \end{pmatrix} + \sqrt{10} \begin{pmatrix} \frac{\sqrt{10}}{10} & 0 & \frac{3\sqrt{10}}{10} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

6. Let $f = \begin{pmatrix} 5 & 4 & 6 & 6 \\ 6 & 1 & 6 & 3 \\ 1 & 2 & 1 & 5 \\ 6 & 4 & 6 & 1 \end{pmatrix}$.

(a) Compute the Haar transform f_{Haar} of f .

(b) Suppose there is only enough capacity to store 10 pixel values of f_{Haar} . Choose 10 entries to keep such that the reconstructed image differs as little as possible in Frobenius norm with the original image, and compute the reconstructed image.

Solution:

$$(a) \tilde{H} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}.$$

$$\begin{aligned} f_{\text{Haar}} &= \tilde{H} f \tilde{H}^T \\ &= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 5 & 4 & 6 & 6 \\ 6 & 1 & 6 & 3 \\ 1 & 2 & 1 & 5 \\ 6 & 4 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 18 & 11 & 19 & 15 \\ 4 & -1 & 5 & 3 \\ -\sqrt{2} & 3\sqrt{2} & 0 & 3\sqrt{2} \\ -5\sqrt{2} & -2\sqrt{2} & -5\sqrt{2} & 4\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 63 & -5 & 7\sqrt{2} & 4\sqrt{2} \\ 11 & -5 & 5\sqrt{2} & 2\sqrt{2} \\ 5\sqrt{2} & -\sqrt{2} & -8 & -6 \\ -8\sqrt{2} & -6\sqrt{2} & -6 & -18 \end{pmatrix} = \begin{pmatrix} \frac{63}{4} & -\frac{5}{4} & \frac{7\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \\ \frac{11}{4} & -\frac{5}{4} & \frac{5\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -2 & -\frac{3}{2} \\ -2\sqrt{2} & -\frac{3\sqrt{2}}{2} & -\frac{3}{2} & -\frac{9}{2} \end{pmatrix}. \end{aligned}$$

(b) Since \tilde{H} is unitary, for any $g \in M_{4 \times 4}(\mathbb{R})$,

$$\|\tilde{H}^T g \tilde{H} - f\|_F = \|\tilde{H}^T (g - f_{\text{Haar}}) \tilde{H}\|_F = \|g - f_{\text{Haar}}\|_F.$$

Hence one should choose to discard the entries with smaller absolute values so as to minimize the Frobenius norm of the difference.

$$\text{Hence the matrix that should be kept is either } f'_{\text{Haar}} = \begin{pmatrix} \frac{63}{4} & 0 & \frac{7\sqrt{2}}{4} & 0 \\ \frac{11}{4} & 0 & \frac{5\sqrt{2}}{4} & 0 \\ \frac{5\sqrt{2}}{4} & 0 & -2 & 0 \\ -2\sqrt{2} & -\frac{3\sqrt{2}}{2} & -\frac{3}{2} & -\frac{9}{2} \end{pmatrix},$$

whose reconstructed image is given by $\tilde{H}^T f'_{\text{Haar}} \tilde{H}$

$$\begin{aligned} &= \frac{1}{16} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 63 & 0 & 7\sqrt{2} & 0 \\ 11 & 0 & 5\sqrt{2} & 0 \\ 5\sqrt{2} & 0 & -8 & 0 \\ -8\sqrt{2} & -6\sqrt{2} & -6 & -18 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\ &= \frac{1}{16} \begin{pmatrix} 84 & 0 & 4\sqrt{2} & 0 \\ 64 & 0 & 20\sqrt{2} & 0 \\ 36 & -12 & -4\sqrt{2} & -18\sqrt{2} \\ 68 & 12 & 8\sqrt{2} & 18\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\ &= \frac{1}{16} \begin{pmatrix} 92 & 76 & 84 & 84 \\ 104 & 24 & 64 & 64 \\ 16 & 32 & 12 & 84 \\ 96 & 64 & 92 & 20 \end{pmatrix} = \begin{pmatrix} \frac{23}{4} & \frac{19}{4} & \frac{21}{4} & \frac{21}{4} \\ \frac{13}{2} & \frac{3}{2} & 4 & 4 \\ 1 & 2 & 3 & \frac{21}{4} \\ 6 & 4 & \frac{23}{4} & \frac{5}{4} \end{pmatrix}; \end{aligned}$$

$$\text{or keep } f''_{\text{Haar}} = \begin{pmatrix} \frac{63}{4} & 0 & \frac{7\sqrt{2}}{4} & 0 \\ \frac{11}{4} & 0 & \frac{5\sqrt{2}}{4} & 0 \\ \frac{5\sqrt{2}}{4} & 0 & -2 & -\frac{3}{2} \\ -2\sqrt{2} & -\frac{3\sqrt{2}}{2} & 0 & -\frac{9}{2} \end{pmatrix},$$

whose reconstructed image is given by $\tilde{H}^T f''_{\text{Haar}} \tilde{H}$

$$\begin{aligned}
&= \frac{1}{16} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 63 & 0 & 7\sqrt{2} & 0 \\ 11 & 0 & 5\sqrt{2} & 0 \\ 5\sqrt{2} & 0 & -8 & -6 \\ -8\sqrt{2} & -6\sqrt{2} & 0 & -18 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{16} \begin{pmatrix} 84 & 0 & 4\sqrt{2} & -6\sqrt{2} \\ 64 & 0 & 20\sqrt{2} & 6\sqrt{2} \\ 36 & -12 & 2\sqrt{2} & -18\sqrt{2} \\ 68 & 12 & 2\sqrt{2} & 18\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{16} \begin{pmatrix} 92 & 76 & 72 & 96 \\ 104 & 24 & 76 & 52 \\ 28 & 20 & 12 & 84 \\ 84 & 76 & 92 & 20 \end{pmatrix} = \begin{pmatrix} \frac{23}{4} & \frac{19}{4} & \frac{9}{2} & 6 \\ \frac{13}{2} & \frac{3}{2} & \frac{19}{4} & \frac{13}{4} \\ \frac{7}{4} & \frac{5}{4} & \frac{3}{4} & \frac{21}{4} \\ \frac{4}{4} & \frac{19}{4} & \frac{23}{4} & \frac{5}{4} \end{pmatrix}.
\end{aligned}$$

7. Let $H_n(t)$ be the n^{th} Haar function, where $n \in \mathbb{N} \cup \{0\}$.

- (a) Write down the definition of $H_n(t)$.
(b) Write down the Haar transform matrix \tilde{H} for 4×4 images.

- (c) Suppose $A = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 \\ 4 & 5 & 5 & 6 \end{pmatrix}$. Compute the Haar transform A_{Haar} of A , and compute

the reconstructed image \tilde{A} after setting the largest entry of A_{Haar} to 0.

Solution:

- (a) $H_0(t) = \mathbf{1}_{[0,1]}$, and for any $p \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{Z} \cap [0, 2^p - 1]$,

$$H_{2^p+n}(t) = 2^{\frac{p}{2}} \left(\mathbf{1}_{[\frac{n}{2^p}, \frac{n+0.5}{2^p})} - \mathbf{1}_{[\frac{n+0.5}{2^p}, \frac{n+1}{2^p})} \right).$$

(b) $\tilde{H} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}$.

- (c)

$$\begin{aligned}
A_{\text{Haar}} &= \tilde{H} A \tilde{H}^T \\
&= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 2 \\ 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 \\ 4 & 5 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 8 & 12 & 12 & 16 \\ -4 & -4 & -4 & -4 \\ -2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} \\ -2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 12 & -2 & -\sqrt{2} & -\sqrt{2} \\ -4 & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Then the modified Haar transform A'_{Haar} is $\begin{pmatrix} 0 & -2 & -\sqrt{2} & -\sqrt{2} \\ -4 & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 \end{pmatrix}$, and thus:

$$\begin{aligned} \tilde{A} &= \tilde{H}^T A'_{\text{Haar}} \tilde{H} \\ &= \frac{1}{4} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & -2 & -\sqrt{2} & -\sqrt{2} \\ -4 & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} -8 & -2 & -\sqrt{2} & -\sqrt{2} \\ 0 & -2 & -\sqrt{2} & -\sqrt{2} \\ 0 & -2 & -\sqrt{2} & -\sqrt{2} \\ 8 & -2 & -\sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} -3 & -2 & -2 & -1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 3 \end{pmatrix}. \end{aligned}$$

8. Suppose the definition of the DFT on $N \times N$ images is changed to

$$\hat{f}(m, n) = DFT(f)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k, l) e^{2\pi j \frac{mk+nl}{N}}.$$

- (a) Does there exist a matrix U such that $\hat{f} = UfU$ for an $N \times N$ image f ? If yes, derive U and check if it is unitary.
(b) Show that the inverse DFT (iDFT) is defined by

$$f(p, q) = iDFT(\hat{f})(p, q) = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \hat{f}(m, n) e^{-2\pi j \frac{pm+qn}{N}}.$$

Solution:

- (a) The matrix U used to calculate the DFT of an $N \times N$ matrix is given by

$$U = (U(x, \alpha))_{0 \leq x, \alpha \leq n}, \text{ where } U(x, \alpha) = \frac{1}{\sqrt{N}} e^{2\pi j \frac{x\alpha}{N}}.$$

To check that U is unitary, we first denote the column of U indexed by α by \vec{u}_α . Then,

- i. For any $0 \leq \alpha \leq N - 1$,

$$\begin{aligned} \langle \vec{u}_\alpha, \vec{u}_\alpha \rangle &= \sum_{x=0}^{N-1} U(x, \alpha) \overline{U(x, \alpha)} \\ &= \sum_{x=0}^{N-1} \frac{1}{\sqrt{N}} e^{2\pi j \frac{x\alpha}{N}} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{x\alpha}{N}} \\ &= N \cdot \frac{1}{N} = 1. \end{aligned}$$

ii. For any $0 \leq \alpha_1, \alpha_2 \leq N_1$ such that $\alpha_1 \neq \alpha_2$,

$$\begin{aligned} \langle \vec{u}_{\alpha_1}, \vec{u}_{\alpha_2} \rangle &= \sum_{x=0}^{N-1} U(x, \alpha_1) \overline{U(x, \alpha_2)} \\ &= \sum_{x=0}^{N-1} \frac{1}{\sqrt{N}} e^{2\pi j \frac{x\alpha_1}{N}} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{x\alpha_2}{N}} \\ &= \frac{1}{N} \sum_{x=0}^{N-1} e^{2\pi j \frac{x(\alpha_1 - \alpha_2)}{N}} \\ &= \frac{1}{N} \cdot N \mathbf{1}_{N\mathbb{Z}}(\alpha_1 - \alpha_2) = 0. \end{aligned}$$

Hence U is unitary.

(b) For any $0 \leq p, q \leq N - 1$,

$$\begin{aligned} iDFT(DFT(f))(p, q) &= \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k, l) e^{2\pi j \frac{m(k-p)+n(l-q)}{N}} \\ &= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k, l) \left[\sum_{m=0}^{N-1} e^{2\pi j \frac{m(k-p)}{N}} \right] \left[\sum_{n=0}^{N-1} e^{2\pi j \frac{n(l-q)}{N}} \right] \\ &= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k, l) \cdot N \mathbf{1}_{N\mathbb{Z}}(k-p) \cdot N \mathbf{1}_{N\mathbb{Z}}(l-q) \\ &= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k, l) \delta(k-p) \delta(l-q) = f(p, q). \end{aligned}$$

9. Let $f = \begin{pmatrix} 3 & 2 & 4 & 4 \\ 4 & -3 & 4 & 0 \\ -2 & -1 & -2 & 3 \\ 4 & 1 & 4 & -2 \end{pmatrix}$.

(a) Compute the discrete Fourier transform \hat{f} of f .

(b) Compute the image reconstructed from \hat{f} after removing frequencies in 3rd row and 3rd column.

Solution:

(a) $U = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$.

$$\hat{f} = UfU$$

$$\begin{aligned} &= \frac{1}{16} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \begin{pmatrix} 3 & 2 & 4 & 4 \\ 4 & -3 & 4 & 0 \\ -2 & -1 & -2 & 3 \\ 4 & 1 & 4 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \\ &= \frac{1}{16} \begin{pmatrix} 9 & -1 & 10 & 5 \\ 5 & 3+4j & 6 & 1-2j \\ -7 & 3 & -6 & 9 \\ 5 & 3-4j & 6 & 1+2j \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \\ &= \frac{1}{16} \begin{pmatrix} 23 & -1+6j & 15 & -1-6j \\ 15+2j & 5-2j & 7-2j & -7+2j \\ -1 & -1+6j & -25 & -1-6j \\ 15-2j & -7-2j & 7+2j & 5+2j \end{pmatrix}. \end{aligned}$$

(b) The submatrix of \hat{f} is

$$\hat{f}' = \frac{1}{16} \begin{pmatrix} 23 & -1+6j & 0 & -1-6j \\ 15+2j & 5-2j & 0 & -7+2j \\ 0 & 0 & 0 & 0 \\ 15-2j & -7-2j & 0 & 5+2j \end{pmatrix},$$

whose reconstructed image is

$$\begin{aligned} (4\bar{U})\hat{f}'(4\bar{U}) &= \frac{1}{16} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \begin{pmatrix} 23 & -1+6j & 0 & -1-6j \\ 15+2j & 5-2j & 0 & -7+2j \\ 0 & 0 & 0 & 0 \\ 15-2j & -7-2j & 0 & 5+2j \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \\ &= \frac{1}{16} \begin{pmatrix} 47 & 49 & 59 & 57 \\ 17 & -17 & 21 & 55 \\ -5 & -27 & -9 & 13 \\ 25 & 39 & 29 & 15 \end{pmatrix}. \end{aligned}$$

10. Let $f, g \in M_{M \times N}(\mathbb{R})$ be periodically extended, please prove $\widehat{f * g} = MN\hat{f} \odot \hat{g}$, where $\hat{f} \odot \hat{g}(m, n) = \hat{f}(m, n)\hat{g}(m, n)$.

Solution:

(a) Refer to DFT of convolution of **Further properties of DFT** in Section 2.3.

(b)

$$\begin{aligned} iDFT(MN\hat{f} \odot \hat{g})(k, l) &= MN \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(m, n)\hat{g}(m, n)e^{2\pi j(\frac{mk}{M} + \frac{nl}{N})} \\ &= \frac{1}{MN} \sum_{m, k', k''=0}^{M-1} \sum_{n, l', l''=0}^{N-1} f(k', l')g(k'', l'')e^{2\pi j(\frac{m(k-k'-k'')}{M} + \frac{n(l-l'-l'')}{N})} \\ &= \sum_{k', k''=0}^{M-1} \sum_{l', l''=0}^{N-1} f(k', l')g(k'', l'')\mathbf{1}_{M\mathbb{Z}}(k - k' - k'')\mathbf{1}_{N\mathbb{Z}}(l - l' - l'') \\ &= \sum_{k', k''=0}^{M-1} \sum_{l', l''=0}^{N-1} f(k', l')g(k'', l'')[\delta(k - k' - k'') + \delta(k - k' - k'' + M)] \\ &\quad [\delta(l - l' - l'') + \delta(l - l' - l'' + N)] \\ &= \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} f(k', l')g(k - k', l - l') = f * g(k, l). \end{aligned}$$

11. Let $f, g \in M_{M \times N}(\mathbb{R})$ be periodically extended, please prove $\widehat{f \odot g} = \hat{f} * \hat{g}$, where $f \odot g(k, l) = f(k, l)g(k, l)$.

Solution:

(a)

$$\widehat{f \odot g}(m, n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k, l)g(k, l)e^{-2\pi j(\frac{mk}{M} + \frac{nl}{N})},$$

whereas

$$\begin{aligned}
\hat{f} * \hat{g}(m, n) &= \sum_{m'=0}^{M-1} \sum_{n'=0}^{N-1} \hat{f}(m', n') \hat{g}(m - m', n - n') \\
&= \frac{1}{M^2 N^2} \sum_{m', k, k'=0}^{M-1} \sum_{n', l, l'=0}^{N-1} f(k, l) e^{-2\pi j(\frac{m'k}{M} + \frac{n'l}{N})} g(k', l') e^{-2\pi j(\frac{(m-m')k'}{M} + \frac{(n-n')l'}{N})} \\
&= \frac{1}{M^2 N^2} \sum_{m', k, k'=0}^{M-1} \sum_{n', l, l'=0}^{N-1} f(k, l) g(k', l') e^{-2\pi j(\frac{mk' + m'(k-k')}{M} + \frac{nl' + n'(l-l')}{N})} \\
&= \frac{1}{MN} \sum_{k, k'=0}^{M-1} \sum_{l, l'=0}^{N-1} f(k, l) g(k', l') e^{-2\pi j(\frac{mk'}{M} + \frac{nl'}{N})} \delta(k - k') \delta(l - l') \\
&= \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k, l) g(k, l) e^{-2\pi j(\frac{mk}{M} + \frac{nl}{N})} = \widehat{f \odot g}(m, n).
\end{aligned}$$

(b)

$$\begin{aligned}
iDFT(\hat{f} * \hat{g})(k, l) &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f} * \hat{g}(m, n) e^{2\pi j(\frac{mk}{M} + \frac{nl}{N})} \\
&= \sum_{m, m'=0}^{M-1} \sum_{n, n'=0}^{N-1} \hat{f}(m', n') \hat{g}(m - m', n - n') e^{2\pi j(\frac{mk}{M} + \frac{nl}{N})} \\
&= \frac{1}{M^2 N^2} \sum_{m, m', k', k''=0}^{M-1} \sum_{n, n', l', l''=0}^{N-1} f(k', l') g(k'', l'') \\
&\quad e^{2\pi j(\frac{m(k-k'') + m'(k''-k')}{M} + \frac{n(l-l'') + n'(l''-l')}{N})} \\
&= \sum_{k', k''=0}^{M-1} \sum_{l', l''=0}^{N-1} f(k', l') g(k'', l'') \\
&\quad \mathbf{1}_{M\mathbb{Z}}(k - k'') \mathbf{1}_{M\mathbb{Z}}(k' - k'') \mathbf{1}_{N\mathbb{Z}}(l - l'') \mathbf{1}_{N\mathbb{Z}}(l' - l'') \\
&= f(k, l) g(k, l).
\end{aligned}$$

12. Let $f \in M_{N \times N}(\mathbb{R})$ be periodically extended, and let $\tilde{f}(k, l) = f(l, -k)$, please prove $\hat{\tilde{f}} = \hat{f}$.

Solution:

(a)

$$\begin{aligned}
\hat{\tilde{f}}(m, n) &= \frac{1}{N^2} \sum_{k, l=0}^{N-1} \tilde{f}(k, l) e^{-2\pi j \frac{mk + nl}{N}} \\
&= \frac{1}{N^2} \sum_{k, l=0}^{N-1} f(l, -k) e^{-2\pi j \frac{mk + nl}{N}},
\end{aligned}$$

whereas

$$\begin{aligned}
\tilde{f}(m, n) &= \hat{f}(n, -m) \\
&= \frac{1}{N^2} \sum_{k,l=0}^{N-1} f(k, l) e^{-2\pi j \frac{nk - ml}{N}} \\
&= \frac{1}{N^2} \sum_{l'=0}^{N-1} \sum_{k'=1-N}^0 f(l', -k') e^{-2\pi j \frac{mk' + nl'}{N}} \\
&= \frac{1}{N^2} \sum_{l'=0}^{N-1} \left(f(l', 0) e^{-2\pi j \frac{nl'}{N}} + \sum_{k'=1-N}^{-1} f(l', -k') e^{-2\pi j \frac{mk' + nl'}{N}} \right) \\
&= \frac{1}{N^2} \sum_{k', l'=0}^{N-1} f(l', -k') e^{-2\pi j \frac{mk' + nl'}{N}} = \hat{f}(m, n).
\end{aligned}$$

(b)

$$\begin{aligned}
iDFT(\tilde{f})(k, l) &= \sum_{m,n=0}^{N-1} \tilde{f}(m, n) e^{2\pi j \frac{mk + nl}{N}} \\
&= \sum_{m,n=0}^{N-1} \hat{f}(n, -m) e^{2\pi j \frac{mk + nl}{N}} \\
&= \sum_{m'=0}^{N-1} \sum_{n'=1-N}^0 \hat{f}(m', n') e^{2\pi j \frac{-n'k + m'l}{N}} \\
&= \sum_{m'=0}^{N-1} \left(\hat{f}(m', 0) e^{2\pi j \frac{m'l}{N}} + \sum_{n'=1-N}^{-1} \hat{f}(m', n' + N) e^{2\pi j \frac{-n'k + m'l}{N}} \right) \\
&= \sum_{m', n'=0}^{N-1} \hat{f}(m', n') e^{2\pi j \frac{m'l - n'k}{N}} = f(l, -k).
\end{aligned}$$

13. Let $f \in M_{M \times N}(\mathbb{R})$ be periodically extended, and let $\tilde{f}(k, l) = f(k - k_0, l - l_0)$ for some $k_0, l_0 \in \mathbb{Z}$, please prove $\hat{\tilde{f}} = e^{-2\pi j (\frac{k_0 m}{M} + \frac{l_0 n}{N})} \hat{f}$.

Solution: WLOG assume $k_0 \in \mathbb{Z} \cap [0, M - 1]$ and $l_0 \in \mathbb{Z} \cap [0, N - 1]$.

(a) Refer to DFT of a shifted image of **Further properties of DFT** in Section 2.3.

(b)

$$\begin{aligned}
iDFT(e^{-2\pi j (\frac{k_0 m}{M} + \frac{l_0 n}{N})} \hat{f})(k, l) &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(m, n) e^{2\pi j (\frac{m(k-k_0)}{M} + \frac{n(l-l_0)}{N})} \\
&= f(k - k_0, l - l_0).
\end{aligned}$$

14. Let $f \in M_{M \times N}(\mathbb{R})$ be periodically extended, and let $\tilde{f}(m, n) = \hat{f}(m - m_0, n - n_0)$ for some $m_0, n_0 \in \mathbb{Z}$, please prove $\tilde{f} = DFT(e^{2\pi j (\frac{km_0}{M} + \frac{ln_0}{N})} f)$.

Solution:

(a)

$$\begin{aligned}
\tilde{f}(m, n) &= \hat{f}(m - m_0, n - n_0) \\
&= \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k, l) e^{-2\pi j (\frac{k(m-m_0)}{M} + \frac{l(n-n_0)}{N})} \\
&= DFT(e^{2\pi j (\frac{m_0 k}{M} + \frac{n_0 l}{N})} f)(m, n).
\end{aligned}$$

(b)

$$\begin{aligned}
iDFT(\tilde{f})(k, l) &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \tilde{f}(m, n) e^{2\pi j(\frac{mk}{M} + \frac{nl}{N})} \\
&= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(m - m_0, n - n_0) e^{2\pi j(\frac{mk}{M} + \frac{nl}{N})} \\
&= \sum_{m'=-m_0}^{M-1-m_0} \sum_{n'=-n_0}^{N-1-n_0} \hat{f}(m', n') e^{2\pi j(\frac{(m'+m_0)k}{M} + \frac{(n'+n_0)l}{N})} \\
&= e^{2\pi j(\frac{m_0k}{M} + \frac{n_0l}{N})} f(k, l).
\end{aligned}$$

15. Please prove that the rank k approximation is the optimal approximation for rank k matrix in sense of Frobenius norm. That is, given a rank r matrix $A \in M_{n \times m}(\mathbb{R})$, for any rank k matrix $B \in M_{n \times m}(\mathbb{R})$, we have

$$\|A - B\|_F \geq \|A - A_k\|_F,$$

where $A_k = \sum_{i=1}^k \sigma_i \vec{u}_i \vec{v}_i^T$ is the rank k approximation of $A = \sum_{i=1}^r \sigma_i \vec{u}_i \vec{v}_i^T$ and $k = 1, 2, \dots, r$.

Solution:

This proof is updated to be the proof in Tutorial 6.

Before proving the desired result, we first prove a result:

$$\|A - A_k\|_2 \leq \|A - B\|_2$$

where $\|\cdot\|_2$ is defined to be

$$\begin{aligned}
\|C\|_2 &:= \sup_{\vec{x} \in \mathbb{R}^N} \frac{\|C\vec{x}\|_2}{\|\vec{x}\|_2} \\
&= \sup_{\substack{\vec{x} \in \mathbb{R}^N \\ \|\vec{x}\|_2=1}} \|C\vec{x}\|_2 \\
&= \sigma_1(C),
\end{aligned}$$

where $\sigma_i(C)$ is the i -th singular value of C .

Note that since B is of rank k , we can rewrite B as XY^T , where $X \in \mathbb{R}^{M \times k}$ and $Y \in \mathbb{R}^{N \times k}$. (You may consider the SVD of $B = PSQ^T$, take X to be the first k columns of PS , and take Y to be the first k columns of Q .)

Let $\vec{v}_1, \dots, \vec{v}_{k+1}$ be the first $k+1$ columns of V . Since the span of these vectors has dimension $k+1$, and the Y^T is of rank k , there must be a non trivial linear combination $\vec{w} = \gamma_1 \vec{v}_1 + \dots + \gamma_{k+1} \vec{v}_{k+1}$ such that $Y^T \vec{w} = \vec{0}$. Assume further that $\|\vec{w}\|_2 = 1$.

Then

$$\begin{aligned}
\|A - B\|_2^2 &= \sup_{\substack{\vec{x} \in \mathbb{R}^N \\ \|\vec{x}\|_2=1}} \|(A - B)\vec{x}\|_2^2 \\
&\geq \|(A - B)\vec{w}\|_2^2 \\
&= \|U\Sigma V^T \vec{w} - XY^T \vec{w}\|_2^2 \\
&= \|\Sigma(\gamma_1 \vec{e}_1 + \dots + \gamma_{k+1} \vec{e}_{k+1})\|_2^2 \\
&= \gamma_1^2 \sigma_1^2 + \dots + \gamma_{k+1}^2 \sigma_{k+1}^2 \\
&\geq \sigma_{k+1}^2 (\gamma_1^2 + \dots + \gamma_{k+1}^2) \\
&= \|A - A_k\|_2^2
\end{aligned}$$

Then back to the proof for the F-norm.

Suppose $A = A' + A''$. Note by the triangle inequality for matrix 2-norm,

$$\sigma_1(M_1 + M_2) \leq \sigma_1(M_1) + \sigma_1(M_2)$$

Then for $i, j \geq 1$,

$$\begin{aligned} \sigma_i(A') + \sigma_j(A'') &= \sigma_1(A' - A'_{i-1}) + \sigma_1(A'' - A''_{j-1}) \\ &\geq \sigma_1(A' - A'_{i-1} + A'' - A''_{j-1}) \\ &\geq \sigma_1(A - A_{i+j-2}) \\ &= \sigma_{i+j-1}(A) \end{aligned}$$

where the second inequality makes use of the fact proved in matrix 2-norm. $A'_{i-1} + A''_{j-1}$ is a matrix of at most rank $i + j - 2$. So the rank $i + j - 2$ approximation A_{i+j-2} of SVD minimize σ_1 .

Then take $A' = A - B, A'' = B$. Choose $1 \leq i \leq \min\{n, m\}, j = k + 1$,

$$\sigma_i(A - B) + \sigma_{k+1}(B) = \sigma_i(A - B) \geq \sigma_{i+k}(A)$$

Then

$$\begin{aligned} \|A - B\|_F^2 &= \sum_{i=1}^{\min\{n, m\}} \sigma_i(A - B)^2 \\ &\geq \sum_{i=1}^{r-k} \sigma_{i+k}(A)^2 \\ &= \sum_{i=k+1}^r \sigma_i(A)^2 \\ &= \|A - A_k\|_F^2 \end{aligned}$$

which is desired.