MATH3360 Mathematical Imaging
Midterm Examination

Name: ________________________ Student ID: ____________________

You have to answer all five questions. The total score is **100**.
Please show your steps unless otherwise stated.

1. Let $\mathcal{O} : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be a linear image transformation.

   (a) Suppose $\mathcal{O}$ is separable and $\mathcal{O}(f) = AfB$ for any $f \in M_{2\times 2}(\mathbb{R})$, where $A, B \in M_{2\times 2}(\mathbb{R})$. Prove that the transformation matrix of $\mathcal{O}$ is $H = B^T \otimes A$.

   (b) Let
$$H = \begin{pmatrix} a & 4 & 4 & 2 \\ 2 & a-2 & b & 3 \\ c & 2 & 12 & 9-d \\ 1 & 3 & d & 9 \end{pmatrix}$$
   be the transformation matrix corresponding to $\mathcal{O}$. Determine suitable $a, b, c, d \in \mathbb{R}$ such that $\mathcal{O}$ is separable and find matrices A and B.

   (c) Suppose $\mathcal{O}$ is defined by convolution and $\mathcal{O}(f) = h * f$ for any $f \in M_{2\times 2}(\mathbb{R})$, where $h \in M_{2\times 2}(\mathbb{R})$. Prove that the transformation matrix H of $\mathcal{O}$ is in the form: $H = \begin{pmatrix} H_1 & H_2 \\ H_2 & H_1 \end{pmatrix}$, where $H_1, H_2 \in M_{2\times 2}(\mathbb{R})$.

2. Let
$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

   (a) Compute SVD of A. Express A as a linear combination of its elementary images.

   (b) Find a rank 2 approximation A_2 such that $\|A_2 - A\|_F = 2$. Please prove your answer with details.

   (c) Using (a), determine the SVD of a distorted image
$$\widetilde{A} = \begin{pmatrix} 1+\epsilon & 2+\tau & 0 \\ 2+\tau & 1+\epsilon & 0 \\ 0 & 0 & 2+\tau \end{pmatrix},$$
   where ϵ and τ are small positive real numbers. Please explain your answer with details.

3. Recall that the 0-th Haar function is
$$H_0(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{otherwise,} \end{cases}$$
   and the other Haar functions are defined by
$$H_{2^p+n}(t) = \begin{cases} \sqrt{2}^p & \text{if } \frac{n}{2^p} \le t < \frac{n+0.5}{2^p} \\ -\sqrt{2}^p & \text{if } \frac{n+0.5}{2^p} \le t < \frac{n+1}{2^p} \\ 0 & \text{otherwise} \end{cases}$$
   for $p = 0, 1, 2, \cdots$ and $n = 0, 1, 2, \cdots, 2^p - 1$.

(a) Give the definition of Haar transformation for 4×4 images.

(b) Suppose

$$A = \begin{pmatrix} 5 & 0 & 0 & 3 \\ 2 & 1 & 1 & 4 \\ 2 & 1 & 1 & 4 \\ 4 & 0 & 0 & 6 \end{pmatrix},$$

please compute the Haar transform A_{Haar} of A.

(c) Suppose A_{Haar} is corrupted by noise, such that

$$\tilde{A}_{\text{Haar}} = A_{\text{Haar}} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}.$$

Let $\widetilde{A}$ be the reconstructed image from $\tilde{A}_{\text{Haar}}$. Discuss which pixels of $\widetilde{A}$ have different intensity values from A. Find the error $||\widetilde{A} - A||_F$.

4. Let $I \in M_{N\times N}(\mathbb{R})$ be a $N \times N$ image, whose indices are taken from 0 to $N-1$. Assuming that I is periodically extended. Let k_0, l_0 be two integers between 0 and $N-1$. Recall that the discrete Fourier transform (DFT) of an $N \times N$ image I is defined as

$$DFT(I)(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} I(k,l) e^{-2\pi\sqrt{-1}(\frac{km+ln}{N})}$$

for all $0 \leq m, n \leq N-1$.

(a) Consider another image I_1 defined by: $I_1(k,l) = I(k-k_0, l-l_0)$ for all $0 \leq k, l \leq N-1$. Write $DFT(I_1)$ in terms of $DFT(I)$.

(b) Consider another image I_2 defined by: $I_2(k,l) = I(-l+l_0, -k+k_0)$ for all $0 \leq k, l \leq N-1$. Write $DFT(I_2)$ in terms of $DFT(I)$.

Please show all your steps clearly (including how the changes of variables are applied, indices are shifted and so on). Missing details will lead to mark deduction.

END OF PAPER