Lecture 8:

$$f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \underbrace{I_{ij}}_{\text{elementary images}}$$

Recall:

- Image decomposition f

  ① Storage saving

  ② Image processing by modifying transformed image (coefficient matrix)
  (e.g. Removing coefficients associated to high-frequency elementary images)

- 2 Separable Image Transformation:

  ① SVD (elementary images not universal and meaningless)

  ② Haar (elementary images universal and meaningful) — unsmooth

Discrete Fourier Transform:

Definition:

The 2D DFT of a $M \times N$ image $g = (g(k, l))_{k,l}$, where $0 \le k \le M-1$, $0 \le l \le N-1$ is defined as:

$$\hat{g}(m, n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k, l)\, e^{-j 2\pi \left(\frac{km}{M} + \frac{ln}{N} \right)}$$

(where $j = \sqrt{-1}$, $e^{j\theta} = \cos\theta + j \sin\theta$)

Remark: The inverse of DFT is given by:

$$g(p, q) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{g}(m, n)\, e^{j 2\pi \left(\frac{pm}{M} + \frac{qn}{N} \right)}$$

DFT of g

(no $\frac{1}{Mn}$!)

(no -ve sign)

Proof of Inverse DFT:

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{j2\pi\left(\frac{pm}{M}+\frac{qn}{N}\right)} \hat{g}(m,n) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{j2\pi\left(\frac{pm}{M}+\frac{qn}{N}\right)} \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km}{M}+\frac{ln}{N}\right)}$$

$$= \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} g(k,l)\, e^{j2\pi\left(\frac{(p-k)m}{M}+\frac{(q-l)n}{N}\right)}$$

$$= \underbrace{\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l) \sum_{m=0}^{M-1} e^{j2\pi\left(\frac{(p-k)m}{M}\right)} \sum_{n=0}^{N-1} e^{j2\pi\left(\frac{(q-l)n}{N}\right)}}_{(*)}$$

Note that:

$$\sum_{m=0}^{M-1} e^{j2\pi\left(\frac{mt}{M}\right)} = \underbrace{\frac{\left[e^{j2\pi\left(\frac{t}{M}\right)}\right]^{M}-1}{e^{j2\pi\left(\frac{t}{M}\right)}-1}}_{\text{if } t\neq 0} = M\,\delta(t) := \begin{cases} M & t=0 \\ 0 & t\neq 0 \end{cases}$$

$\therefore$ (*) becomes:

$$\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l)\, M\,\delta(p-k)\, N\,\delta(q-l) = g(p,q).$$

DFT in Matrix form

Theorem: Consider a $N \times N$ image g, the DFT of g can be written as:

$$\hat{g} = \mathcal{U} g \mathcal{U} \quad \text{(DFT in matrix form)}$$

where $\mathcal{U} = (U_{kl})_{0 \le k, l \le N-1} \in M_{N\times N}$ and $\mathcal{U}_{kl} = \frac{1}{N} e^{-j\frac{2\pi kl}{N}}$.

Proof: Need to check $\hat{g}(k,l) = (\mathcal{U} g \mathcal{U})(k,l)$ (k-th row, l-th col of $\mathcal{U} g \mathcal{U}$)

$$\text{LHS} = \hat{g}(k,l) = \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} g(m,n)\, e^{-j2\pi\left(\frac{km}{N} + \frac{ln}{N}\right)}$$

$$\text{RHS}: \quad \mathcal{U} g \mathcal{U} = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} g(m,n) \begin{pmatrix} | \\ \vec{u}_m \\ | \end{pmatrix} \begin{pmatrix} - \vec{u}_n - \end{pmatrix}$$

$$\mathcal{U} = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_N \\ | & | & & | \end{pmatrix}, \qquad \mathcal{U} = \begin{pmatrix} - \vec{u}_1 - \\ - \vec{u}_2 - \\ \vdots \\ - \vec{u}_N - \end{pmatrix}$$

$$\vec{u}_m = \begin{pmatrix} u_{0m} \\ u_{1m} \\ \vdots \\ u_{N-1\,m} \end{pmatrix} = \frac{1}{N} \begin{pmatrix} e^{-j\frac{2\pi(0)m}{N}} \\ \vdots \\ e^{-j\frac{2\pi(k)m}{N}} \\ \vdots \\ e^{-j\frac{2\pi(N-1)m}{N}} \end{pmatrix}$$

$$\mathcal{U} g \mathcal{U}(k,l) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} g(m,n) \frac{e^{-j2\pi\frac{km}{N}}}{N} \cdot \frac{e^{-j2\pi\frac{ln}{N}}}{N}$$

$$= \text{LHS}$$

$$\vec{u}_n = (u_{n0}, u_{n1}, \ldots, u_{nN}) = \frac{1}{N}\left(e^{-j\frac{2\pi(0)(n)}{N}}, \ldots, e^{-j\frac{2\pi(l)(n)}{N}}, \ldots\right)$$

<u>Theorem</u>: $U^* U = \frac{1}{N} I$ where $U^* = (\overline{U})^T$ (conjugate transpose)

$U U^* = \frac{1}{N} I$.

$\therefore U^{-1} = (NU)^*$

$(\overline{a+jb} = a-jb)$

$(\overline{e^{j\theta}} = \overline{\cos\theta + j\sin\theta} = \cos\theta - j\sin\theta$
$= e^{-j\theta})$

<u>Proof</u>: Consider $(U^* U)(k,l)$ (k-th row, l-th col of U^*U)

$$(U^* U)(k,l) = \left(\underbrace{\overline{\qquad\qquad}}_{k\text{-th row of } U^*}\right)\Bigg(\Big|\Big)\ l\text{-col of } U$$

Let $U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_N \\ | & | & & | \end{pmatrix}$

$\overline{U} = \begin{pmatrix} - & \overline{\vec{u}_1^T} & - \\ - & \overline{\vec{u}_2^T} & - \\ & \vdots & \\ - & \overline{\vec{u}_N^T} & - \end{pmatrix}$

$(\overline{U})^* = \begin{pmatrix} - & \overline{(\vec{u}_1^T)} & - \\ & \vdots & \\ - & \overline{(\vec{u}_N^T)} & - \end{pmatrix}$

$$= \overline{(\vec{u}_k^T)}\, \vec{u}_l$$

$$= \left(\overline{\frac{e^{-j2\pi\frac{k(0)}{N}}}{N}}, \ldots, \overline{\frac{e^{-j2\pi\frac{k\alpha}{N}}}{N}}, \ldots, \overline{\frac{e^{-j2\pi\frac{k(N-1)}{N}}}{N}}\right) \begin{pmatrix} \frac{e^{-j2\pi\frac{l(0)}{N}}}{N} \\ \vdots \\ \frac{e^{-j2\pi\frac{l(\alpha)}{N}}}{N} \\ \vdots \\ \frac{e^{-j2\pi\frac{l(N-1)}{N}}}{N} \end{pmatrix}$$

$$= \sum_{\alpha=0}^{N-1} \frac{e^{j2\pi\frac{k\alpha}{N}}}{N} \frac{e^{-j2\pi\frac{l\alpha}{N}}}{N} = \sum_{\alpha=0}^{N-1} \frac{e^{-j2\pi(\alpha)(l-k)}}{N^2}$$

$$= \frac{1}{N}\delta(l-k)$$

$$\therefore \quad \mathcal{U}^*\mathcal{U}(k,l) = \begin{cases} \frac{1}{N} & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

$$\Rightarrow \quad \mathcal{U}^*\mathcal{U} = \frac{1}{N} I$$

Similarly, $\mathcal{U}\mathcal{U}^* = \frac{1}{N} I$

Image decomposition by DFT

Suppose $\hat{g} = DFT(g) = \mathcal{U} g \mathcal{U}$

Then: $\mathcal{U}\mathcal{U}^* = \frac{1}{N} I = \mathcal{U}^*\mathcal{U}$

$\therefore \quad g = (N\mathcal{U})^* \hat{g} (N\mathcal{U})^*$

$\therefore \quad g = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \hat{g}_{kl} \vec{w}_k \vec{w}_l^T$ Elementary image of DFT

where $\vec{w}_k = k^{th}$ col of $(N\mathcal{U})^*$