Lecture 7

Haar transformation

(From now on, we assume all images $f = (f_{ij})_{\substack{0 \le i \le N-1 \\ 0 \le j \le N-1}}$

Definition: (Haar functions) The Haar functions are defined recursively as follows

$$H_0(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

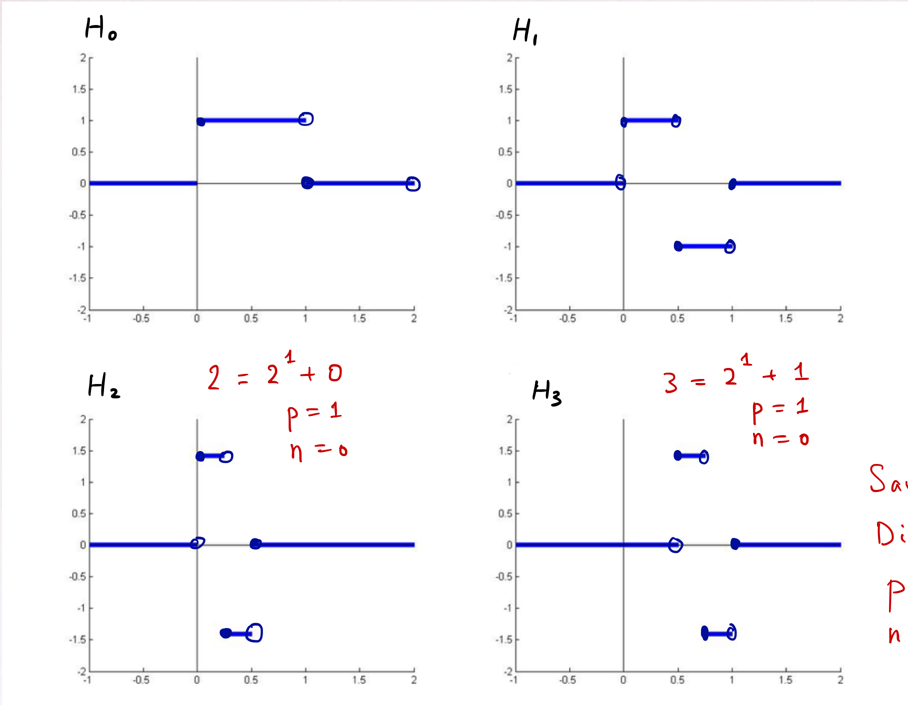
$$H_1(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1/2 \\ -1 & \text{if } 1/2 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$H_{2^p+n} \equiv \begin{cases} \sqrt{2}^p & \text{if } \frac{n}{2^p} \le t < \frac{n+0.5}{2^p} \\ -\sqrt{2}^p & \text{if } \frac{n+0.5}{2^p} \le t < \frac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases}$$

where $p = 1, 2, \dots;\ n = 0, 1, 2, \dots, 2^p - 1$

Remark: If p is larger, H_{2^p+n} is compactly supported in a smaller region.

Examples of Haar functions:



Same wavefront

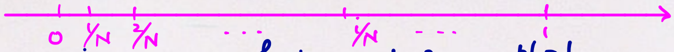
Different locations

$p \leftrightarrow$ wavefront

$n \leftrightarrow$ location

Definition (Discrete Haar Transform)

The Haar Transform of a $N \times N$ image is done by dividing $[0,1]$ into partitions.



Let $H(k,i) \equiv H_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \ldots, N-1$.

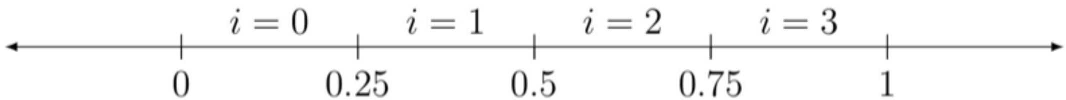
We obtain the Haar Transform matrix: $\tilde{H} \equiv \frac{1}{\sqrt{N}} H$ where $H \equiv \left(H(k,i)\right)_{0 \le k, i \le N-1}$

The Haar Transform of $f \in M_{n \times n}$ is defined as:

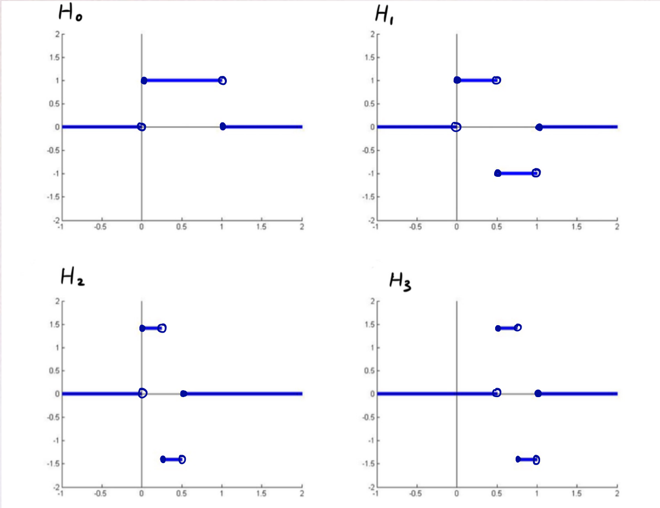
$$g = \tilde{H} f \tilde{H}^T.$$

Example Compute the Haar Transform matrix for a 4×4 image.

Solution: Divide $[0, 1]$ into 4 portions:



Need to check:



$$\begin{pmatrix} H(0,0) & H(0,1) & H(0,2) & H(0,3) \\ H(1,0) & H(1,1) & H(1,2) & H(1,3) \\ H(2,0) & H(2,1) & H(2,2) & H(2,3) \\ H(3,0) & H(3,1) & H(3,2) & H(3,3) \end{pmatrix}$$

$$\|$$

We get that:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \quad \text{and} \quad \tilde{H} = \frac{1}{\sqrt{4}} H = \frac{1}{2} H$$

Easy to check that $\tilde{H}^T \tilde{H} = I$.

$$\|$$

$$\begin{pmatrix} H_0\left(\frac{0}{4}\right) & H_0\left(\frac{1}{4}\right) & H_0\left(\frac{2}{4}\right) & H_0\left(\frac{3}{4}\right) \\ H_1\left(\frac{0}{4}\right) & H_1\left(\frac{1}{4}\right) & H_1\left(\frac{2}{4}\right) & H_1\left(\frac{3}{4}\right) \\ & & & \\ & & & \end{pmatrix}$$

Example 2 Compute the Haar Transform of

$$f = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Solution:

$$g = \tilde{H} f \tilde{H}^T = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

More zeros

Example 3 Suppose g in Example 2 is changed to:

$$g = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Reconstruct the original image.

Solution:

$$f = \tilde{H}^T g \tilde{H} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Localized error

Remark:

1. Haar Transform usually produces coefficient matrix with more zeros!

2. Localized error in coefficient matrix causes localized error in the reconstructed image

Elementary images under Haar transform:

Using Haar transform, f can be written as:

$$f = \tilde{H}^T g \tilde{H}$$

g ← transformed image

Let $\tilde{H} = \begin{pmatrix} - \vec{h}_1^T - \\ - \vec{h}_2^T - \\ \vdots \\ - \vec{h}_N^T - \end{pmatrix}$. Then: $f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \vec{h}_i \vec{h}_j^T$

$\vec{h}_i \vec{h}_j^T = I_{ij}^H$

I_{ij}^T = elementary images under Haar Transform.

e.g. For 8 x 8 images, we have 64 = 8 x 8 elementary images:

$$\begin{matrix} \vec{h}_0 \vec{h}_0^T & \vec{h}_0 \vec{h}_1^T & \cdots & \vec{h}_0 \vec{h}_7^T \\ \vec{h}_1 \vec{h}_0^T & \vec{h}_1 \vec{h}_1^T & \cdots & \vec{h}_1 \vec{h}_7^T \\ \vdots & \vdots & & \vdots \\ \vec{h}_7 \vec{h}_0^T & \vec{h}_7 \vec{h}_1^T & & \vec{h}_7 \vec{h}_7^T \end{matrix}$$

(each $\vec{h}_i \vec{h}_j^T \in M_{8\times 8}(\mathbb{R})$)

Discrete Fourier Transform:

Definition: The 1D discrete Fourier Transform (DFT) of a function $f(k)$, defined at discrete points $k = 0, 1, 2, \ldots, N-1$ is defined as:

$$\hat{f}(m) = \frac{1}{N} \sum_{k=0}^{N-1} f(k)\, e^{-j\frac{2\pi m k}{N}} \quad \left(\text{where } j = \sqrt{-1},\ e^{j\theta} = \cos\theta + j\sin\theta\right)$$

The 2D DFT of a $M \times N$ image $g = (g(k,l))_{k,l}$, where $0 \le k \le M-1$, $0 \le l \le N-1$ is defined as:

$$\hat{g}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km}{M} + \frac{ln}{N}\right)}$$

Remark: The inverse of DFT is given by:

$$g(p,q) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{g}(m,n)\, e^{j2\pi\left(\frac{pm}{M} + \frac{qn}{N}\right)}$$

(no $\frac{1}{Mn}$!)

DFT of g

(no -ve sign)

DFT in Matrix form

Theorem: Consider a $N \times N$ image g, the DFT of g can be written as:

$$\hat{g} = \mathcal{U}\, g\, \mathcal{U} \quad \text{(DFT in matrix form)}$$

where $\mathcal{U} = (\mathcal{U}_{kl})_{0 \le k, l \le N-1} \in M_{N \times N}$ and $\mathcal{U}_{kl} = \frac{1}{N} e^{-j\frac{2\pi k l}{N}}$.

$$\mathcal{U} = \frac{1}{N}\begin{pmatrix} e^{-j\frac{2\pi(0)(0)}{N}} & e^{-j\frac{2\pi(0)(1)}{N}} & \cdots \\ & e^{-j\frac{2\pi(1)(1)}{N}} & \\ & & \ddots \end{pmatrix}$$

$\therefore$ $\mathcal{U}$ is Symmetric

$\mathcal{U}$ is "almost" Orthogonal

Example Find the DFT of the following 4×4 image

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Solution

The matrix U is given by:

$$U = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

$$U = \left(U_{kl} \right)_{k,l} = \left(\frac{e^{-j2\pi\left(\frac{kl}{4}\right)}}{4} \right)$$

$$\therefore \text{DFT of } g = \hat{g} = UgU = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$