

Lecture 6:

Recap

Image decomposition

Image decomposition based on Singular Value Decomposition (SVD)

Definition: (SVD) For any $g \in M_{m \times n}$, the singular value decomposition (SVD) of g is a matrix factorization: $g = U \Sigma V^T$, where $U \in M_{m \times m}$, $V \in M_{n \times n}$ are orthogonal, $\Sigma \in M_{m \times n}$ is a diagonal matrix ($\Sigma_{ij} = 0$ if $i \neq j$) with diagonal entries given by: $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ with $r \leq \min(m, n)$. ($U U^T = U^T U = I$; $V V^T = V^T V = I$)

singular values

Theorem: The rank of g is given by the number of non-zero singular values.

Remark: Consider an image g . Let $g = U \Sigma V^T$ be the SVD of g (with diagonal entries of Σ given by $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$)

1. Note that $g = U \Sigma V^T = \sum_{i=1}^r \sigma_i U \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & & & \ddots \\ & & & & 0 \end{pmatrix}^{i\text{th}} V^T = \sum_{i=1}^r \sigma_i \vec{u}_i \vec{v}_i^T$
 $\vec{u}_i \vec{v}_i^T$ is called the eigen-image of g under SVD.

2. For $N \times N$ image, the required storage is:

$$\left(\underbrace{N}_{\vec{u}_i} + \underbrace{N}_{\vec{v}_i} + \underbrace{1}_{\sigma_i} \right) \times \underbrace{r}_{r\text{ terms}} = (2N+1)r$$

Definition: For any k ($0 \leq k \leq r$), we define

$$g_k = \sum_{j=1}^k \sigma_j \vec{u}_j \vec{v}_j^T \quad (\text{rank-}k \text{ approximation of } g)$$

$$\| \underbrace{U \begin{pmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \dots & & \\ & & & \sigma_k & \\ & & & & \dots \end{pmatrix} V^T}_{\text{Rank-}k \text{ !!}}$$

Rank - k !!

How to compute SVD

Let $A \in M_{m \times n}$

Step 1: Find eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$
and orthonormal eigenvectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$
of $A^T A \in M_{n \times n}$ (with $\|\vec{v}_j\| = 1, j=1, \dots, n$)

[Recall: $(A^T A) \vec{v}_j = \lambda_j \vec{v}_j$]

Step 2: Define: $\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \\ \text{---} & & & & \text{---} \\ & & & & & 0 \end{pmatrix} \in M_{m \times n}$
Add zero rows if $m > n$

Step 3: For non-zero $\sigma_1, \sigma_2, \dots, \sigma_r$,
let $\vec{u}_1 = \frac{A \vec{v}_1}{\sigma_1}, \vec{u}_2 = \frac{A \vec{v}_2}{\sigma_2}, \dots, \vec{u}_r = \frac{A \vec{v}_r}{\sigma_r}$

Step 4: Extend $\{\vec{u}_1, \dots, \vec{u}_r\}$ to the basis
 $\{\vec{u}_1, \dots, \vec{u}_r, \dots, \vec{u}_m\}$ of \mathbb{R}^m .

Step 5: Let:

$$U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \dots & \vec{u}_m \\ | & | & & | \end{pmatrix} \in M_{m \times m}$$

$$V = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{pmatrix} \in M_{n \times n}$$

Then: $A = U \Sigma V^T$

Example 2.1: Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}.$$

We have

$$A^T A = \begin{pmatrix} 9 & 8 \\ 8 & 9 \end{pmatrix}.$$

Now, $\text{eig}(A^T A)$ are 17 and 1, and so $\sigma_1 = \sqrt{17}$, $\sigma_2 = 1$ and

$$\tilde{\Sigma} = \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \Sigma = \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Moreover,

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This gives

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

Since

$$\frac{A\vec{v}_1}{\sigma_1}$$

$$\sigma_1 \vec{u}_1 = A\vec{v}_1,$$

$$u_i = \frac{A\vec{v}_i}{\sigma_i}$$

we have

$$\vec{u}_1 = \frac{1}{\sqrt{17}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{34}} \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}.$$

$$\frac{A\vec{v}_2}{\sigma_2}$$

u

Similarly, we have

$$\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The matrix U is, therefore, given by

$$U = \begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} \\ \frac{4}{\sqrt{34}} & 0 \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} \end{pmatrix} \mathbf{u}_3$$

for some vector \mathbf{u}_3 orthonormal to both \mathbf{u}_1 and \mathbf{u}_2 . One possibility is

$$(\text{cross-product}) \vec{u}_3 = \frac{1}{\sqrt{17}} \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}.$$

Finally, the SVD of A is given by

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \\ \frac{4}{\sqrt{34}} & 0 & \frac{-3}{\sqrt{17}} \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \end{pmatrix} \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}.$$

$$\begin{aligned}
\text{Thus, } \|D\|_F^2 &= \sum_m \sum_n D_{mn}^2 \\
&= \sum_m \sum_n \left(\sum_{i=k+1}^r \sigma_i^2 u_{im}^2 v_{in}^2 + 2 \sum_{i=k+1}^r \sum_{\substack{j=k+1 \\ j \neq i}}^r \sigma_i \sigma_j u_{im} v_{in} u_{jm} v_{jn} \right) \\
&= \sum_{i=k+1}^r \sigma_i^2 \sum_m u_{im}^2 \sum_n v_{in}^2 + 2 \sum_{i=k+1}^r \sum_{\substack{j=k+1 \\ j \neq i}}^r \sigma_i \sigma_j \sum_m u_{im} u_{jm} \sum_n v_{in} v_{jn} \\
&= \sum_{i=k+1}^r \sigma_i^2 = \lambda_i
\end{aligned}$$

- Remark:
- To approximate an image using SVD, arrange the eigenvalues λ_i in decreasing order and remove the last few terms in $\sum_{i=1}^r \sigma_i \vec{u}_i \vec{v}_i^T$
 - rank- k approximation is the optimal approximation using k -terms (in term of F-norm) (or with rank- k image)