Convolution

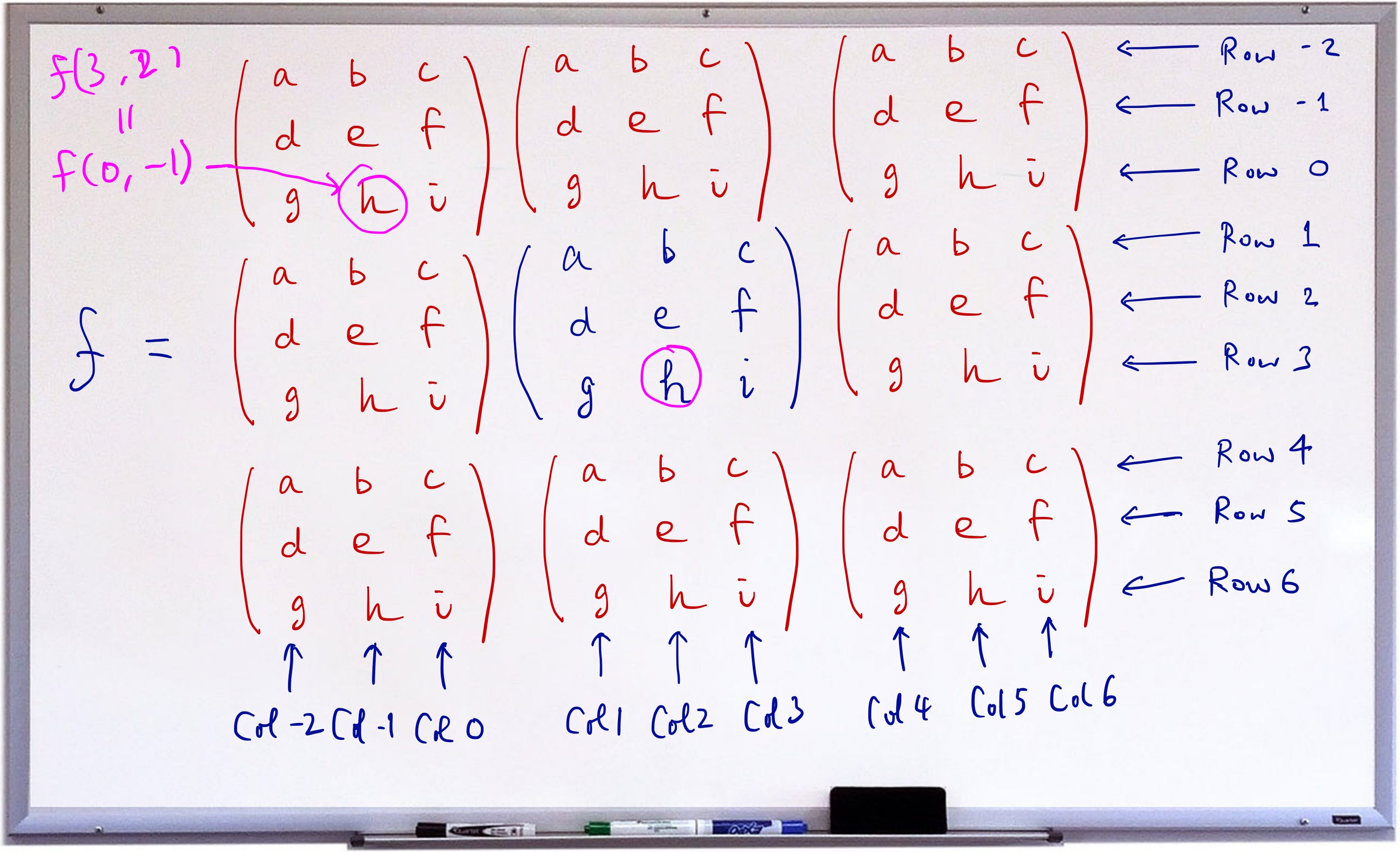
Definition: Consider $k \in M_{N\times N}(\mathbb{R})$ and $f \in M_{N\times N}(\mathbb{R})$. Assume k and f are periodically extended. That is:

$$k(x,y) = k(x+pN, y+qN)$$
$$f(x,y) = f(x+pN, y+qN)$$

where p, q are integers.

The convolution $k*f$ of k and f is a $N\times N$ matrix defined as:

$$k*f(\alpha,\beta) = \sum_{x=1}^{N}\sum_{y=1}^{N} k(x,y)\, f(\alpha-x, \beta-y) \quad \text{for } 1 \le \alpha, \beta \le N$$



f(3, 2)
=
f(0, -1)
f =
a b c
d e f
g h i
Row -2
Row -1
Row 0
Row 1
Row 2
Row 3
Row 4
Row 5
Row 6
Col -2
Col -1
Col 0
Col 1
Col 2
Col 3
Col 4
Col 5
Col 6

Example: Let $k = \begin{pmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{pmatrix}$ and $f \in M_{3\times 3}(\mathbb{R})$. Find $k * f(2,2)$

$$k * f(2,2) = \sum_{x=1}^{3} \sum_{y=1}^{3} k(x,y)\, f(2-x, 2-y)$$

$$= \frac{1}{9} f(1,1) + \frac{1}{9} f(1,0) + \frac{1}{9} f(1,-1) + \frac{1}{9} f(0,1) + \frac{1}{9} f(0,0) + \frac{1}{9} f(0,-1)$$

$$+ \frac{1}{9} f(-1,1) + \frac{1}{9} f(-1,0) + \frac{1}{9} f(-1,-1)$$

$$= \frac{f(1,1) + f(1,2) + f(1,3) + f(2,1) + f(2,2) + f(2,3) + f(3,1) + f(3,2) + f(3,3)}{9}$$

(Averaging the intensity values in the neighborhood of $f(2,2)$)

Remark: Averaging is commonly used in image processing, which is related to convolution.

Theorem: Let $k \in M_{N\times N}(\mathbb{R})$. Define $\mathcal{U} : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ by:

$$\mathcal{U}(f) = k * f \quad \text{for all } f \in M_{N\times N}(\mathbb{R}).$$

Then: $\mathcal{U}$ is linear.

Pf: Followed from the definition of convolution.

Theorem: Let $k \in M_{N\times N}(\mathbb{R})$ and $f \in M_{N\times N}(\mathbb{R})$. Then: $k * f = f * k$.

Proof: Assuming $\alpha, \beta > 1$,

$$k*f(\alpha,\beta) = \sum_{x=1}^{N}\sum_{y=1}^{N} k(x,y)\, f(\alpha-x, \beta-y)$$

$$= \sum_{\tilde{x}=\alpha-N}^{\alpha-1}\sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-\tilde{x}, \beta-\tilde{y})\, f(\tilde{x},\tilde{y}) \quad (\text{let } \tilde{x} = \alpha - x,\ \tilde{y} = \beta - y)$$

$$\overset{??}{=} \sum_{\tilde{x}=1}^{N}\sum_{\tilde{y}=1}^{N} k(\alpha-\tilde{x}, \beta-\tilde{y})\, f(\tilde{x},\tilde{y}) = f*k(\alpha,\beta).$$

$$\sum_{\tilde{x}=\alpha-N}^{\alpha-1} \sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-\tilde{x}, \beta-\tilde{y})\, f(\tilde{x}, \tilde{y})$$

$$= \sum_{\tilde{x}=\alpha-N}^{0} \left(\sum_{\tilde{y}=\beta-N}^{\beta-1} \underbrace{k(\alpha-\tilde{x}, \beta-\tilde{y})}_{=\, k(\alpha-(\tilde{x}+N),\, \beta-\tilde{y})}\, \underbrace{f(\tilde{x}, \tilde{y})}_{=\, f(\tilde{x}+N,\, \tilde{y})} \right) + \sum_{\tilde{x}=1}^{\alpha-1} \left(\sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-\tilde{x}, \beta-\tilde{y})\, f(\tilde{x}, \tilde{y}) \right)$$

(periodic extension)

$$= \sum_{\tilde{x}=\alpha}^{N} \left(\sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-\tilde{x}, \beta-\tilde{y})\, f(\tilde{x}, \tilde{y}) \right) + \sum_{\tilde{x}=1}^{\alpha-1} \left(\sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-\tilde{x}, \beta-\tilde{y})\, f(\tilde{x}, \tilde{y}) \right)$$

$$= \sum_{\tilde{x}=1}^{N} \left(\sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-\tilde{x}, \beta-\tilde{y})\, f(\tilde{x}, \tilde{y}) \right) = \sum_{\tilde{x}=1}^{N} \sum_{\tilde{y}=1}^{N} k(\alpha-\tilde{x}, \beta-\tilde{y})\, f(\tilde{x}, \tilde{y})$$

$$= f * k(\alpha, \beta)$$

The case when $\alpha=1$ or $\beta=1$ can be shown similarly.

Geometric meaning of convolution

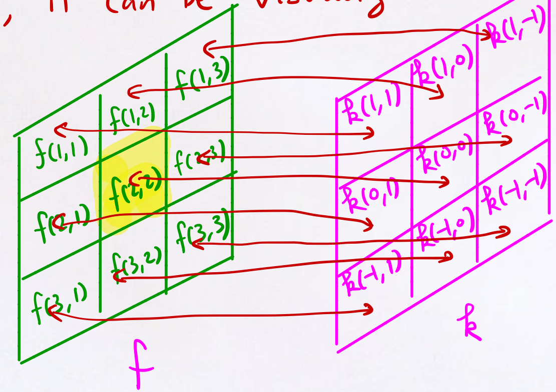
Consider $k \in M_{3\times3}(\mathbb{R})$ and $f \in M_{3\times3}(\mathbb{R})$.

Consider: $k * f(2,2) = \sum_{x=1}^{3}\sum_{y=1}^{3} k(2-x,2-y)f(x,y)$

$= k(1,1)f(1,1) + k(1,0)f(1,2) + k(1,-1)f(1,3) + k(0,1)f(2,1) + k(0,0)f(2,2)$

$+ k(0,-1)f(2,3) + k(-1,1)f(3,1) + k(-1,0)f(3,2) + k(-1,-1)f(3,3)$

Geometrically, it can be visualized as dot product:



Overlay k onto f and take dot product.

__Example__: Let $\mathcal{O}: M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ be a linear image transformation defined by: $\mathcal{O}(f)(\alpha,\beta) = f(\alpha+1,\beta) + 2f(\alpha,\beta) - 2f(\alpha-1,\beta) + f(\alpha,\beta+1) - 2f(\alpha,\beta-1)$ for all $1 \leq \alpha,\beta \leq N$ and $f \in M_{N\times N}(\mathbb{R})$. Show that $\mathcal{O}$ can be expressed in terms of a convolution.

Suppose $\mathcal{O}(f) = k * f$ for some $k \in M_{N\times N}(\mathbb{R})$. Then,

Then: $$\mathcal{O}(f)(\alpha,\beta) = \sum_{x=1}^{N}\sum_{y=1}^{N} k(\alpha-x,\beta-y)\, f(x,y)$$

$$= \ldots + \overset{1}{\overset{\shortparallel}{k(-1,0)}} f(\alpha+1,\beta) + \overset{2}{\overset{\shortparallel}{k(0,0)}} f(\alpha,\beta) + \overset{-2}{\overset{\shortparallel}{k(1,0)}} f(\alpha-1,\beta) + \overset{1}{\overset{\shortparallel}{k(0,-1)}} f(\alpha,\beta+1)$$
$$+ \underset{-2}{\underset{\shortparallel}{k(0,1)}} f(\alpha,\beta-1) + \ldots$$

We set $k(-1,0) = k(N-1,N) = 1$, $k(0,0) = k(N,N) = 2$, $k(1,0) = k(1,N) = -2$
$k(0,-1) = k(N,N-1) = 1$, $k(0,1) = k(N,1) = -2$ and $k(x,y) = 0$ otherwise.

Then: $\mathcal{O}(f) = k * f$.

Definition: The point spread function $h^{\alpha,\beta}(x,y)$ of a linear image transformation is called shift-invariant if there exists a function $\tilde{h}$ such that

$$h^{\alpha,\beta}(x,y) = \tilde{h}(\alpha - x, \beta - y)$$

for all $1 \le x, y, \alpha, \beta \le N$.

Remark: Given $k \in M_{N\times N}(\mathbb{R})$. Let $\mathcal{O}$ be a linear image transformation defined by: $\mathcal{O}(f) = k * f$ for all $f \in M_{N\times N}(\mathbb{R})$.

Then: the point spread function of $\mathcal{O}$ is shift-invariant.

Let $g = \mathcal{O}(f)$

$$g(\alpha,\beta) = \mathcal{O}(f)(\alpha,\beta) = \sum_{x=1}^{N}\sum_{y=1}^{N} \underbrace{k(\alpha - x, \beta - y)}_{= h^{\alpha,\beta}(x,y)} f(x,y)$$