Lecture 24: Chan-Vese Segmentation

Implicit representation of curves

Let $I : \Omega \to \mathbb{R}$ be an image. Let $\varphi : \Omega \to \mathbb{R}$ be a real-valued function defined on Ω. Suppose Ω_0 is the object in Ω, we may assume that $\varphi(z) > 0$ inside Ω_0 and $\varphi(z) < 0$ outside Ω_0 (i.e. in $\Omega \setminus \Omega_0$), Then, the zero-level set $\varphi^{-1}(\{0\}) = \{\vec{p} \in \Omega : \varphi(\vec{p}) = 0\}$ is a curve in Ω, which reoresents the boundary of the object. This is called the **level set representation** of the curve and φ is called the level set function.

Computation of the level set function

Let Ω_0 be the object. Want: $\varphi : \Omega \to \mathbb{R}$ representing Ω_0. Suppose we know the inside and outside regions of Ω_0. Let $d(\vec{x}, \partial\Omega_0) = \inf_{\vec{y} \in \partial\Omega_0} ||\vec{x} - \vec{y}||$ = minimal distance from $\vec{x}$ to the boundary of Ω_0, $\partial\Omega_0$. We can define the level set function as:

$$\varphi(\vec{x}) = \begin{cases} d(\vec{x}, \partial\Omega_0) \text{ if } \vec{x} \in \text{ inside} \\ -d(\vec{x}, \partial\Omega_0) \text{ if } \vec{x} \in \text{ outside} \end{cases}$$

Obviously, $\varphi^{-1}(\{0\}) = \partial\Omega_0$.

Remark.

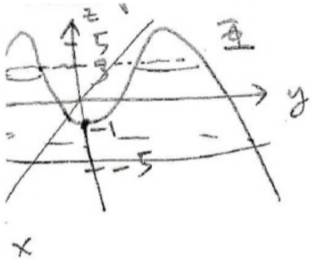
- *Working with the level set function for curve evolution increases the dimension of the problem (from 1D to 2D).*
- *If $\varphi : \mathbb{R}^3 \to \mathbb{R}$, then $\varphi^{-1}(\{0\})$ is a surface embedded in $\mathbb{R}^3$ (implicit function theorem).*

Example 1. Consider: $\varphi(\vec{x}) = 2 - (x^2 + y^2)$. Then, $\varphi^{-1}(\{0\})$ = circle centred at the origin with radius $= \sqrt{2}$.

To expand the circle, we consider: $\varphi(\vec{x}) = (2 + \varepsilon) - (x^2 + y^2)$. Then, $\varphi_\varepsilon^{-1}(\{0\})$ = circle centred at the origin with radius $= \sqrt{2+\varepsilon}$ $(> \sqrt{2})$.

Remark. *By modifying φ, we can evolve a curve.*

Example 2. Consider a function $\Phi : \mathbb{R}^2 \to \mathbb{R}$ which looks like:



Let $\varphi_K(x, y) = K - \Phi(x, y)$. Then,

$$\begin{aligned}
\varphi_{-5}^{-1}(\{0\}) &= \text{ one simple closed curve} \\
\varphi_{-1}^{-1}(\{0\}) &= \text{ two simple closed curves touching each other} \\
\varphi_{3}^{-1}(\{0\}) &= \text{ two separated simple closed curve}
\end{aligned}$$

Remark. *Level set method allows topological change. Therefore, level set based segmentation algorithm can evolve one closed curve to several closed curves capturing the boundaries of multiple objects.*

Level set method for image segmentation

Goal: Find a level set function φ that minimizes:

$$E(\varphi) = \underbrace{F_1(\varphi)}_{\text{smoothness}} + \underbrace{F_2(\varphi)}_{\text{data ficlelity}}$$

Assumption:

1. $F_1(\varphi)$ = smoothness term. Smoothness is achieved by minimizing the length of $\varphi^{-1}(\{0\})$.
2. $F_2(\varphi)$ = want $\{\varphi > 0\}$ matches with the object in the image. We assume the image is piecewise constant. Therefore, each object has uniform (constant) image intensity.

Geometric quantities obtained from level set representation

Outward unit normal: $\dfrac{\nabla\phi(\vec{x})}{|\nabla\phi(\vec{x})|}$, $\vec{x} \in \Gamma = \phi^{-1}(\{0\})$

Curvature of contour: $\kappa(\vec{x}) = div\left(\dfrac{\nabla\phi(\vec{x})}{|\nabla\phi(\vec{x})|}\right)$; $\vec{x} \in \Gamma$

Curve length element ds: $ds = \delta(\phi(\vec{x}))|\nabla\phi(\vec{x})|d\vec{x}$, where δ = delta function.

Chan-Vese Segmentation model/ Active Contour without edge

Minimize: $E(\varphi) = F_1(\varphi) + F_2(\varphi)$

For $F_1(\varphi)$, we choose it as:

$$\begin{aligned} Length(\Gamma) &= \int_\Gamma d\sigma \\ &= \int \delta(\varphi(\vec{x}))|\nabla\phi(\vec{x})|d\vec{x} \\ &= \int |\nabla H(\varphi(\vec{x}))|d\vec{x} \end{aligned}$$

where $H(x) = \text{Heaviside function} = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ and $\nabla H(\varphi(\vec{x}))$ means the gradient of $H \circ \varphi : \mathbb{R}^2 \to \mathbb{R}$ at $\vec{x}$. (Coarea formula)

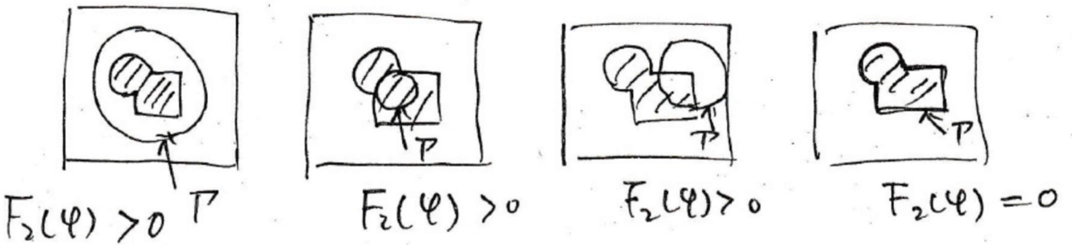
Usually, a smooth approxiamtion of $H(x)$ is used (e.g. $H_k(x) = \frac{1}{1+e^{-2kx}}$)

For $F_2(\varphi)$, we use:

$$\int_{\text{inside } \Gamma} |I(x,y) - c_1|^2 \, dx \, dy + \int_{\text{outside } \Gamma} |I(x,y) - c_2|^2 \, dx \, dy$$

where c_1 = average I inside Γ, c_2 = average I outside Γ.

Observe:



Therefore, the overall Chan-Vese Segmentation model: Find level set function φ and intensities c_1, c_2 such that they minimize:

$$E(\varphi, c_1, c_2) = \int_\Omega |\nabla H(\varphi(\vec{x}))| d\vec{x} + \lambda \int_{\{\varphi > 0\}} |I(x,y) - c_1|^2 \, dx \, dy + \lambda \int_{\{\varphi < 0\}} |I(x,y) - c_2|^2 \, dx \, dy$$

Again, $\nabla H(\varphi(\vec{x}))$ means the gradient of $H \circ \varphi : \mathbb{R}^2 \to \mathbb{R}$ at $\vec{x}$.

If φ is fixed, the contour $\Gamma = \varphi^{-1}\{0\}$ is fixed. Then,

$$0 = \frac{\partial}{\partial c_1} E(\Gamma, c_1, c_2) = -2 \int_{\{\varphi>0\}} (I(x,y) - c_1)\ dx\ dy$$

Therefore, $c_1 = \dfrac{\int_{\{\varphi>0\}} I(x,y)\ dx\ dy}{\int_{\{\varphi>0\}}\ dx\ dy} = \dfrac{\int_\Omega H(\varphi) I(x,y)\ dx\ dy}{\int_\Omega H(\varphi)\ dx\ dy}$.

Similarly, $c_2 = \dfrac{\int_{\{\varphi<0\}} I(x,y)\ dx\ dy}{\int_{\{\varphi<0\}}\ dx\ dy} = \dfrac{\int_\Omega (1 - H(\varphi)) I(x,y)\ dx\ dy}{\int_\Omega (1 - H(\varphi))\ dx\ dy}$

Now, fixing c_1, c_2, we proceed to look for a time-dependent level set function $\varphi = \varphi(t, x, y)$ that minimizes $E(\varphi)$.

Using gradient descent method, we get

$$\frac{\partial \varphi}{\partial t} = \underbrace{\delta(\varphi)}_{\text{smooth approx.}} \left[\underbrace{\nabla \cdot \left(\frac{\nabla \varphi}{|\nabla \varphi|} \right)}_{\text{curvature}} - \lambda (I - c_1)^2 + \lambda (I - c_2)^2 \right],$$

where $\varphi(0, x, y) = \varphi_0(x, y) =$ initial prescribed level set function.

We leave it as an exercise to check the formula.