Lecture 2:

Recap:

Definition: (Linear image transformation)

An image transformation $\mathcal{O} : M_{N \times N}(\mathbb{R}) \to M_{N \times N}(\mathbb{R})$ is linear if it satisfies:

$$\mathcal{O}(af + bg) = a\,\mathcal{O}(f) + b\,\mathcal{O}(g) \text{ for all } f, g \in M_{N \times N}(\mathbb{R}),\ a, b \in \mathbb{R}.$$

Point Spread Function

Take $f \in \mathcal{I} = M_{N\times N}(\mathbb{R})$.

Let
$$f = \begin{pmatrix} f(1,1) & \cdots & & f(1,N) \\ f(2,1) & & & f(2,N) \\ \vdots & & f(x,y) & \vdots \\ f(N,1) & \cdots & & f(N,N) \end{pmatrix} = \sum_{x=1}^{N}\sum_{y=1}^{N} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & f(x,y) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} = \sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y) \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

(x, y) entry

Consider a linear image transformation $\mathcal{O}: M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$.

Let $g = \mathcal{O}(f)$. Then:

$$g(\alpha,\beta) = \left[\sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y)\, \mathcal{O}\left(\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}\right)\right]_{\alpha,\beta}$$

$$= \sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y)\, h^{\alpha,\beta}(x,y)$$

where

$$h^{\alpha,\beta}(x,y) = \left[\mathcal{O}(P_{xy})\right]_{\alpha,\beta} \; ; \; P_{xy} = \begin{array}{cc} \begin{array}{ccccc} & & y^{th} & & \\ & & \downarrow & & \end{array} & \\ \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} & \begin{array}{c} \\ \\ \leftarrow x^{th} \\ \\ \\ \end{array} \end{array}$$

Example: Let $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Define: $\theta : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by:

$$\theta(f) = Af \quad \text{for all } f \in M_{2\times 2}(\mathbb{R}).$$

Consider: $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then: $f = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

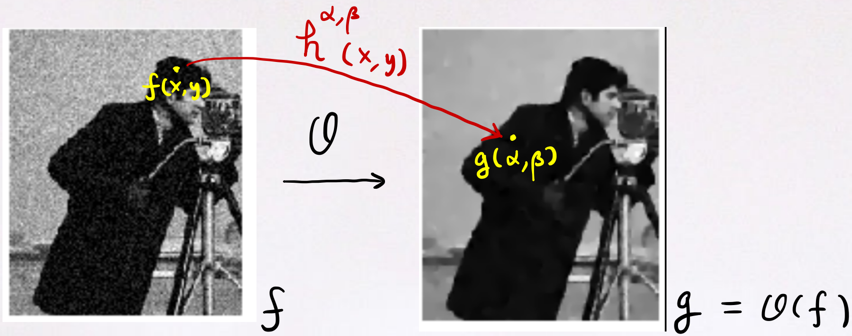
Then: $g = \theta(f) = a\,\underbrace{\theta\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)}_{\text{"}A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} + b\,\theta\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) + c\,\theta\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) + d\,\theta\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$

$$= a\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \underbrace{\theta(f)(1,2)}_{g(1,2)} = a\,\underbrace{\theta\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)(1,2)}_{h^{1,2}(1,1)}{}^{=0} + b\,\underbrace{\theta\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)(1,2)}_{h^{1,2}(1,2)}{}^{=1} + c\,\overbrace{\theta\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)(1,2)}^{h^{1,2}(2,1)}{}_{=0}$$

$$+ d\,\underbrace{\theta\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)(1,2)}_{h^{1,2}(2,2)} = 0$$

Remark: $h^{\alpha,\beta}(x,y)$ determines how much the pixel value of f at (x,y) influences the pixel value of g at (α,β).



Definition: (Point spread function)

$h^{\alpha,\beta}(x,y)$ is usually called the point spread function (PSF)

Separable linear image transformation

Definition: An image transformation $\mathcal{O}: M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ is said to be separable if there exists matrices $A \in M_{N\times N}(\mathbb{R})$ and $B \in M_{N\times N}(\mathbb{R})$ such that: $\mathcal{O}(f) = AfB$ for all $f \in M_{N\times N}(\mathbb{R})$.

Example:

- $\mathcal{O}(f) = \alpha f$ for $\alpha \in \mathbb{R}$

$$= (\alpha I) f (I)$$

- Discrete Fourier Transform is separable
- Discrete Haar Wavelet Transform is separable.

Remark: Separable image transformation is linear.

Theorem: Let $\mathcal{O}$ be a separable image transformation given by : $\mathcal{O}(f) = AfB$ for all $f \in M_{N\times N}(\mathbb{R})$, where $A, B \in M_{N\times N}(\mathbb{R})$. Then, the point spread function of $\mathcal{O}$ is given by :

$$h^{\alpha,\beta}(x,y) = A(\alpha,x)\,B(y,\beta)$$

where $A(\alpha,x)$ is the (α,x) entry of A, $B(y,\beta)$ is the (y,β) entry of B.

Proof: Let $g = \mathcal{O}(f) = AfB$. Then, the (α,β) entry of g is given by :

$$g(\alpha,\beta) = \sum_{x=1}^{N} A(\alpha,x)\,(fB)(x,\beta)$$

$$A(gB)(\alpha,\beta) = \begin{pmatrix} \cancel{A(\alpha,1)\ A(\alpha,2)\ \dots\ A(\alpha,N)} \end{pmatrix} \begin{pmatrix} (gB)(1,\beta) \\ (gB)(2,\beta) \\ \vdots \\ (gB)(N,\beta) \end{pmatrix}$$

$$= \sum_{x=1}^{N} A(\alpha,x) \sum_{y=1}^{N} f(x,y)\,B(y,\beta)$$

$$= \sum_{x=1}^{N}\sum_{y=1}^{N} \underbrace{A(\alpha,x)\,B(y,\beta)}_{= h^{\alpha,\beta}(x,y)}\, f(x,y)$$

Periodic extension of images

Let $f \in M_{N\times N}(\mathbb{R})$ be an image. We say f is periodically extended if $f(x,y) = f(x+pN,\, y+qN)$ where p, q are integers.

(So, f can now be defined on the entire plane $-\infty < x, y < \infty$)

For example, $f(N+10,\, 2N+5) = f(10, 5)$

Geometrically, suppose $f \in M_{3\times 3}(\mathbb{R})$

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$$
\begin{array}{cccl}
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\begin{array}{l} \leftarrow \text{row } -2 \\ \leftarrow \text{row } -1 \\ \leftarrow \text{row } 0 \end{array} \\
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\begin{array}{l} \leftarrow \text{row } 1 \\ \\ \end{array} \\
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\end{array}
$$

$\therefore\ f(1, 5) = b$

$f(-2, 0) = c$

Convolution

Definition: Consider $k \in M_{N \times N}(\mathbb{R})$ and $f \in M_{N \times N}(\mathbb{R})$. Assume k and f are periodically extended. That is:

$$k(x,y) = k(x+pN, y+qN)$$
$$f(x,y) = f(x+pN, y+qN)$$

where p, q are integers.

The convolution $k * f$ of k and f is a $N \times N$ matrix defined as:

$$k * f(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} k(x,y) f(\alpha - x, \beta - y) \quad \text{for } 1 \le \alpha, \beta \le N$$

<u>Example</u>: Let $k = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $f = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. Find $k * f \in M_{2\times 2}(\mathbb{R})$.

The $(1,1)$ entry of $k * f$ is defined as:

$$k*f\,(1,1) = \sum_{x=1}^{2}\sum_{y=1}^{2} k(x,y)\, f(1-x,\ 1-y)$$

$$= k(1,1)\,\underset{\overset{\shortparallel}{f(2,2)}}{f(0,0)} + k(1,2)\,\underset{\overset{\shortparallel}{f(2,1)}}{f(0,-1)} + k(2,1)\,\underset{\overset{\shortparallel}{f(1,2)}}{f(-1,0)} + k(2,2)\,\underset{\overset{\shortparallel}{f(1,1)}}{f(-1,-1)}$$

$$= (1)(1) + (2)(1) + (3)(2) + (4)(1) = 13.$$

Similarly, $k*f\,(1,2) = 14$, $k*f\,(2,1) = 11$, $k*f\,(2,2) = 12$.

$$\therefore\quad k*f = \begin{pmatrix} 13 & 14 \\ 11 & 12 \end{pmatrix}.$$