Lecture 21:

Image processing by minimization
(Variational approach)

1. Consider a minimization model (usually in continuous sense)
2. Derive a PDE related to minimization model
3. Discretize the PDE to get a linear system.

e.g. Total-variation (TV) denoising model. (ROF)
(Rudin - Osher - Fatemi)

2D integration by part formula

Let $f : [a,b]\times[a,b] \to \mathbb{R}$ and $g : [a,b]\times[a,b] \to \mathbb{R}$.

Assume
$$f(a,y) = f(b,y) = f(x,a) = f(x,b) = 0.$$
$$g(a,y) = g(b,y) = g(x,a) = g(x,b) = 0.$$

Then:
$$\int_a^b\int_a^b \nabla f(x,y)\cdot\nabla g(x,y)\,dx\,dy = -\int_a^b\int_a^b \Delta f(x,y)\,g(x,y)\,dx\,dy$$

Proof:
$$\int_a^b\int_a^b \underbrace{\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}}_{\nabla f\cdot\nabla g}\,dx\,dy = -\int_a^b\int_a^b \left(\frac{\partial^2 f}{\partial x^2}\right) g\,dx\,dy + \cancelto{0}{\int_a^b \left(\frac{\partial f}{\partial x}\right) g\Big|_{x=a}^{x=b}\,dy}$$
$$-\int_a^b\int_a^b \left(\frac{\partial^2 f}{\partial y^2}\right) g\,dx\,dy + \cancelto{0}{\int_a^b \frac{\partial f}{\partial y}\, g\Big|_{y=a}^{y=b}\,dx}$$
$$= -\int_a^b\int_a^b \underbrace{\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)}_{\Delta f} g\,dx\,dy$$

Also,

$$\int_a^b \int_a^b \big(k(x,y)\,\nabla f(x,y)\big) \cdot \nabla g(x,y)\,dx\,dy = -\int_a^b \int_a^b \nabla \cdot \big(k(x,y)\,\nabla f(x,y)\big)\, g(x,y)\,dx\,dy$$

where $k : [a,b] \times [a,b] \to \mathbb{R}$.

<u>Proof:</u>

$$\int_a^b \int_a^b \left[k(x,y) \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + k(x,y) \frac{\partial f}{\partial y} \frac{\partial g}{\partial y} \right] dx\,dy$$

$$= -\int_a^b \int_a^b \frac{\partial}{\partial x}\left(k(x,y) \frac{\partial f}{\partial x}\right) g \; dx\,dy + \int_a^b \cancelto{0}{k(x,y) \frac{\partial f}{\partial x}\, g \Big|_{x=a}^{x=b}} \, dy$$

$$- \int_a^b \int_a^b \frac{\partial}{\partial y}\left(k(x,y) \frac{\partial f}{\partial y}\right) g \; dx\,dy + \int_a^b \cancelto{0}{k(x,y) \frac{\partial f}{\partial y}\, g \Big|_{y=a}^{y=b}} \, dx$$

$$= -\int_a^b \int_a^b \underbrace{\left[\frac{\partial}{\partial x}\left(k(x,y) \frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(k(x,y) \frac{\partial f}{\partial y}\right) \right]}_{\nabla \cdot (k(x,y)\, \nabla f)} g \; dx\,dy$$

In general, we have:

Useful Tool: (Integration by part)

$$\nabla \cdot (V_1(x,y), V_2(x,y)) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y}$$

$$\int_\Omega \nabla f \cdot \nabla g \, dx\,dy = -\int_\Omega \big(\underbrace{\nabla \cdot}_{\text{divergence}} (\nabla f)\big) g \, dx\,dy + \int_{\partial\Omega} g \, (\nabla f \cdot \vec{n}) \, ds$$

where $\vec{n} = (n_1, n_2)$ = outward normal on the boundary.

or more generally,

$$\int_\Omega K(x,y) \nabla f(x,y) \cdot \nabla g(x,y) \, dx\,dy = -\int_\Omega \nabla \cdot \big(K(x,y) \nabla f(x,y)\big) g(x,y) \, dx\,dy + \int_{\partial\Omega} g(x,y) \big(K(x,y) \nabla f(x,y) \cdot \vec{n}\big) ds$$

Another useful fact:

If: $\int_\Omega T(x,y)\, v(x,y)\, dx\, dy = 0$ for all $v(x,y)$

then, we can conclude $T(x,y) = 0$ in Ω

Example: Suppose we have the following integral equation:

$$\int_a^b \int_a^b \left(f(x,y) - g(x,y)\right) v(x,y)\, dx\,dy + \int_a^b \int_a^b \nabla \cdot \nabla f(x,y)\; k(x,y)\; v(x,y)\, dx\, dy = 0$$

for all $v(x,y)$. Then: we have:

$$\int_a^b \int_a^b \left[\left(f(x,y) - g(x,y)\right) + k(x,y)\, \nabla \cdot \nabla f(x,y)\right] v(x,y)\, dx\, dy = 0$$

for all $v(x,y)$

We can conclude: $\left(f(x,y) - g(x,y)\right) + k(x,y)\, \nabla \cdot \nabla f(x,y) = 0$

for all $(x,y) \in [a,b] \times [a,b]$

Image denoising by solving PDE (derived from energy minimisation problem)

Consider the harmonic - L2 minimization model:

$$\text{minimize} \quad E(f) = \underbrace{\int_a^b\int_a^b}_{\text{image domain}} (f(x,y) - \underbrace{g(x,y)}_{\text{Observed}})^2 \, dx\,dy + \int_a^b\int_a^b \underbrace{|\nabla f|^2}_{\text{Smoothness of } f} \, dx\,dy$$

(Look for (continuous) image f)

Assume that $f(x,y) = g(x,y) = 0$ on the boundary of $[a,b]\times[a,b]$.

Suppose f minimizes $E(f)$. Let $v : [a,b]\times[a,b] \to \mathbb{R}$ such that $v(x,y) = 0$ on the boundary of $[a,b]\times[a,b]$.

Consider $f^\varepsilon = f + \varepsilon v : [a,b]\times[a,b] \to \mathbb{R}$, which is another image with $f^\varepsilon(x,y) = 0$ on the boundary of $[a,b]\times[a,b]$.

$$f^\varepsilon(x,y) = \underset{0}{\underset{\shortparallel}{f(x,y)}} + \varepsilon\, \underset{0}{\underset{\shortparallel}{v(x,y)}} = 0 \quad \text{on } \partial\big([a,b]\times[a,b]\big).$$

Consider $S: \mathbb{R} \to \mathbb{R}$ defined by:

$$S(\varepsilon) \stackrel{\text{def}}{=} E(f^{\varepsilon}) = E(f + \varepsilon v).$$

Note that $S(0) = E(f) =$ minimum of E. Thus, S attains its minimum at $\varepsilon = 0$.

$$\therefore \frac{dS}{d\varepsilon}(0) = 0.$$

Now, $$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\varepsilon) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E(f + \varepsilon v) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left(\int_a^b \int_a^b (f(x,y) + \varepsilon v(x,y) - g(x,y))^2 \, dx\,dy + \int_a^b \int_a^b |\nabla (f + \varepsilon v)(x,y)|^2 \, dx\,dy \right)$$

$$|\nabla (f + \varepsilon v)|^2 = (\nabla f + \varepsilon \nabla v) \cdot (\nabla f + \varepsilon \nabla v) = \nabla f \cdot \nabla f + 2\varepsilon \nabla f \cdot \nabla v + \varepsilon^2 \nabla v \cdot \nabla v = |\nabla f|^2 + 2\varepsilon \nabla f \cdot \nabla v + \varepsilon^2 |\nabla v|^2$$

$$= \int_a^b \int_a^b 2(f(x,y) + \varepsilon v(x,y) - g(x,y))\, v(x,y) \Big|_{\varepsilon=0} dx\,dy + \int_a^b \int_a^b (2 \nabla f \cdot \nabla v + 2\varepsilon |\nabla v|^2)\Big|_{\varepsilon=0} dx\,dy$$

$$= \int_a^b \int_a^b 2(f(x,y) - g(x,y))\, v(x,y) \, dx\,dy + \int_a^b \int_a^b 2 \nabla f(x,y) \cdot \nabla v(x,y) \, dx\,dy$$

$$\therefore \; S'(0) = 0 = 2\int_a^b\int_a^b \big(f(x,y) - g(x,y)\big)\, v(x,y)\, dx\, dy + 2\int_a^b\int_a^b \left(\frac{\partial f}{\partial x}(x,y)\, \frac{\partial v}{\partial x}(x,y) + \frac{\partial f}{\partial y}(x,y)\, \frac{\partial v}{\partial y}(x,y)\right) dx\, dy$$

for all $v(x,y)$. — $(*)$

If we can formulate $(*)$ in the form:

$$\int_a^b\int_a^b T(x,y)\, v(x,y) = 0 \quad \text{for all } v(x,y),$$

then we can conclude that $T(x,y) = 0$ in $[a,b] \times [a,b]$.

<u>Remark:</u>

- First term is in the form $\int_a^b\int_a^b T(x,y)\, v(x,y)$
- Second term is NOT.

  Need to reformulate the second term.

  Strategy: integration by part.

Second term: $\int_a^b \int_a^b \nabla f(x,y) \nabla v(x,y)\, dx\, dy = -2 \int_a^b \int_a^b \Delta f(x,y)\, v(x,y)\, dx\, dy.$

All together, we have

$$0 = S'(0) = \int_a^b \int_a^b 2\big(f(x,y) - g(x,y)\big)\, v(x,y)\, dx\, dy - 2\int_a^b \int_a^b \Delta f(x,y)\, v(x,y)\, dx\, dy$$

$$\therefore \int_a^b \int_a^b \Big(2\big(f(x,y) - g(x,y)\big) - 2\,\Delta f(x,y)\Big)\, v(x,y)\, dx\, dy = 0 \text{ for all } v(x,y).$$

We conclude:

$$2\big(f(x,y) - g(x,y)\big) - 2\,\Delta f(x,y) = 0 \text{ for } (x,y) \in [a,b] \times [a,b]$$

or

$$f(x,y) - g(x,y) - \Delta f(x,y) = 0$$ (converse of Laplacian masking !!)

Remark: More generally, if we do not enforce $f(x,y) = g(x,y) = 0$ on the boundary of the image domain Ω. Then, we do not enforce

$$v(x,y) = 0 \quad \text{on} \quad \partial\Omega.$$

In this case, we have:

$$0 = S'(0) = \int_\Omega 2\big(f(x,y) - g(x,y)\big)\, v(x,y)\, dx\,dy + \int_\Omega 2\,\nabla f(x,y) \cdot \nabla v(x,y)\, dx\,dy$$

$$= \int_\Omega 2\big(f(x,y) - g(x,y)\big)\, v(x,y)\, dx\,dy - \int_\Omega 2\,\Delta f(x,y)\, v(x,y)\, dx\,dy + \int_{\partial\Omega} \big(2\,\nabla f(x,y) \cdot \vec{n}(x,y)\big)\, v(x,y)\, ds$$

Overall, we get: $\displaystyle\int_\Omega (f - g - \Delta f)\, v\, dx\,dy - \int_{\partial\Omega} (\nabla f \cdot \vec{n})\, v\, ds = 0$ for all v

We conclude:
$$\begin{cases} f - g - \Delta f = 0 & \text{in } \Omega \\ \nabla f \cdot \vec{n} = 0 & \text{in } \partial\Omega \end{cases} \quad \text{(PDE)}$$

<u>Example:</u> Consider an image denoising model to find $f: \underbrace{[a,b]\times[a,b]}_{\Omega} \to \mathbb{R}$ that minimizes:

$$E(f) = \int_a^b\int_a^b (f(x,y)-g(x,y))^2 \, dxdy + \int_a^b\int_a^b |\nabla f(x,y)|^4 \, dxdy.$$

Suppose f minimizes $E(f)$. Assume $f(x,y) = g(x,y) = 0$ for all $(x,y) \in \partial D$. Find a partial differential equation that f must satisfy.

<u>Solution:</u> Suppose f minimizes $E(f)$. For any $v: D \to \mathbb{R}$ such that $v(x,y) = 0$ on ∂D, we have:

$$\begin{cases} f^\varepsilon \overset{\text{def}}{=} f + \varepsilon v \text{ is an image with} \\ f^\varepsilon(x,y) = f(x,y) + \varepsilon v(x,y) = 0 \text{ on } \partial D. \end{cases}$$

Consider $S : \mathbb{R} \to \mathbb{R}$ where $S(\varepsilon) \stackrel{\text{def}}{=} E(f^{\varepsilon}) = E(f + \varepsilon v)$

Then, $S(0) = E(f) =$ minimum of E. Thus, S attains minimum at $\varepsilon = 0$.

$\therefore$ $\left.\dfrac{dS}{d\varepsilon}\right|_{\varepsilon=0} = 0$ for all $v : D \to \mathbb{R}$

Now,

$$0 = \left.\frac{dS}{d\varepsilon}\right|_{\varepsilon=0} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\left(\int_D \big(f(x,y) + \varepsilon v(x,y) - g(x,y)\big)^2\,dx\,dy + \int_D |\nabla(f+\varepsilon v)(x,y)|^4\,dx\,dy\right)$$

$$\left(|\nabla f + \varepsilon \nabla v|^2\right)^2$$

$$= \big((\nabla f + \varepsilon \nabla v)\cdot(\nabla f + \varepsilon \nabla v)\big)^2$$

$$= \left(\nabla f\cdot\nabla f + 2\varepsilon \nabla f\cdot\nabla v + \varepsilon^2 \nabla v\cdot\nabla v\right)^2$$

$$= \left(|\nabla f|^2 + 2\varepsilon\, \nabla f\cdot\nabla v + \varepsilon^2 |\nabla v|^2\right)^2$$

$$\therefore \quad 0 = \frac{dS}{d\varepsilon}(0) = \int_D 2\left(f(x,y) + \varepsilon v(x,y) - g(x,y)\right) v(x,y) \Big|_{\varepsilon=0} dxdy$$

$$+ \int_D 2\left(|\nabla f|^2 + 2\varepsilon \nabla f \cdot \nabla v + \varepsilon^2 |\nabla v|^2\right)\left(2 \nabla f \cdot \nabla v + 2\varepsilon |\nabla v|^2\right)\Big|_{\varepsilon=0} dxdy$$

$$\Rightarrow \quad 0 = \int_D 2\left(f(x,y) - g(x,y)\right) v(x,y)\, dxdy + \int_D 4\left(|\nabla f|^2\right) \nabla f \cdot \nabla v \, dxdy$$

$$= \int_D 2\left(f(x,y) - g(x,y)\right) v(x,y)\, dxdy - \int_D \left(4\, \nabla \cdot \left(|\nabla f|^2 \nabla f\right)(x,y)\right) v(x,y)\, dxdy$$

$$+ \int_{\partial D} \left(\cancelto{0}{4\left(|\nabla f|^2\right) \nabla f(x,y) \cdot \vec{n}(x,y)}\right) \underbrace{v(x,y)}_{=0} dxdy$$

All together, we have:

$$0 = \int_D \left(2\left(f(x,y) - g(x,y)\right) - 4\,\nabla\cdot\left(|\nabla f(x,y)|^2\,\nabla f(x,y)\right)\right) v(x,y)\,dx\,dy$$

for all $v(x,y)$.

We can conclude that:

$$f(x,y) - g(x,y) - 4\,\nabla\cdot\left(|\nabla f(x,y)|^2\,\nabla f(x,y)\right) = 0 \text{ in } D.$$

(Partial differential equation)

Remark: • Anisotropic diffussion is related to minimizing:

$$E(f) = \int_\Omega K(x,y)\,|\nabla f(x,y)|^2\,dx\,dy$$

• Energy minimization approach for solving imaging problem is called the variational image processing!

$$E(f+\varepsilon v) = \int k(x,y)\,|\nabla f + \varepsilon \nabla v|^2\,dx\,dy$$

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E(f+\varepsilon v) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_\Omega k(x,y)\,(\nabla f + \varepsilon \nabla v)\cdot(\nabla f + \varepsilon \nabla v)$$

$$= \int_\Omega k(x,y)\,(2\nabla f \cdot \nabla v)$$

$$= -\int_\Omega \left(\nabla \cdot (k(x,y)\nabla f)\right) v + \int_{\partial\Omega} \left(k(x,y)\nabla f \cdot \vec{n}\right) v$$