Image denoising by PDE

Consider the PDE:

$$(*) \quad \frac{\partial I(x,y,t)}{\partial t} = t\left[\frac{\partial^2 I(x,y,t)}{\partial x^2} + \frac{\partial^2 I(x,y,t)}{\partial y^2}\right] = \underset{\substack{\parallel \\ t \Delta I}}{t\,\nabla\cdot(\nabla I)}$$

$\left(\nabla\cdot = \text{divergence} \,;\, \nabla\cdot(v_1, v_2) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}\right)$ $\left(\nabla I = \left(\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y}\right)\right)$

Then: $g(x,y,t) = \frac{1}{2\pi t^2} e^{-(x^2+y^2)/2t^2}$ satisfies $(*)$.

Given an image I (Assume I is continuously defined on the 2D domain $[a,b] \times [a,b]$)

Note that:

Gaussian filter = convolution of I with the Gaussian function $g(x,y,t)$:

$$\tilde{I}(x,y,t) \overset{def}{=} I * g(x,y,t) \approx \int_a^b \int_a^b g(u,v;t)\, I(x-u,y-v)\, du dv$$

(Analogous to discrete convolution)

We can regard: $\tilde{I}(x,y,0) = I(x,y)$

Then: $\tilde{I}(x,y,t)$ satisfies:

$$(*) \quad \frac{\partial I(x,y,t)}{\partial t} = t\left[\frac{\partial^2 I(x,y,t)}{\partial x^2} + \frac{\partial^2 I(x,y,t)}{\partial y^2}\right]$$

Now, we consider a simplified PDE:

$$(*) \quad \frac{\partial I(x,y;t)}{\partial t} = \Delta I(x,y,t) = \nabla \cdot \left(\nabla I(x,y,t) \right)$$

Discretize it: (we consider a sequence of images: $\underset{\underset{I}{\shortparallel}}{I^0}, I^1, I^2, \dots, I^n, \dots$) that Satisfies (*))

$$\underset{\substack{\uparrow \\ \text{image at time } n+1}}{I^{n+1}}(x,y) - \underset{\substack{\uparrow \\ \text{image at time } n}}{I^n}(x,y) = \Delta I^n(x,y) = p * I^n \qquad p = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\therefore \quad I^{n+1}(x,y) = I^n(x,y) + \Delta I^n(x,y)$$

Remark: PDE gives us a recursive relationship to obtain a Sequence of images $\underset{\underset{I}{\shortparallel}}{I^0}, I^1, \dots, I^n, \dots$, which are smoother and smoother.

Recall:

In the discrete case, we can estimate:

$$\Delta f(x,y) \approx f(x+1,y) + f(x,y+1) + f(x-1,y) + f(x,y-1) - 4f(x,y)$$

Taylor expansion:

$$\frac{\partial^2 f(x,y)}{\partial x^2} \approx \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2} \xrightarrow{\text{Put } h=1} \nabla^2 f(x,y) = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)(x,y)$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) \approx \frac{f(x,y+h) - 2f(x,y) + f(x,y-h)}{h^2}$$

More generally, $\Delta f = p * f \longleftarrow$ discrete convolution

where

$$p = \underset{y=0}{\left(\begin{array}{ccccc} 0 & & \cdots & & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & 1 & -4 & 1 & \vdots \\ & & 1 & & \\ 0 & & \cdots & & 0 \end{array}\right)} x=0$$

$$f(x+h,y) = f(x,y) + \frac{\partial f}{\partial x}(x,y)h + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x,y)h^2 \quad — ①$$

$$f(x-h,y) = f(x,y) - \frac{\partial f}{\partial x}(x,y)h + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x,y)h^2 \quad — ②$$

① + ② gives

Anisotropic diffusion for edge-preserving Image denoising

General "diffusion" eqn :

$$\frac{\partial I(x,y;t)}{\partial t} = \nabla \cdot \left(\underbrace{K(x,y)} \nabla I(x,y;t) \right)$$

- controls the rate of diffusion
- Smaller K = smaller diffusion at (x,y)

Edge detector : Edge of an image can be detected by: $|\nabla I(x,y)|$.

If (x,y) is on the edge, $|\nabla I(x,y)|$ is big.

If (x,y) is in the interior region, $|\nabla I(x,y)| \approx 0$

$\therefore$ Suitable $K(x,y)$ to preserve edge: 1. $K(x,y) = \dfrac{1}{|\nabla I(x,y)| + \varepsilon^2}$

Avoid singularity

2. $K(x,y) = e^{-|\nabla I(x,y;\sigma)|/b}$

∴ The denoising problem can be written as:

$$\frac{\partial I(x,y;t)}{\partial t} = \nabla \cdot \left(e^{-\frac{|\nabla I(x,y;t)|}{b}} \nabla I(x,y;t) \right)$$

In the discrete case, we solve:

$$I^{n+1}(x,y) - I^{n}(x,y) = \mathcal{D}_1 \left(e^{-\frac{|\nabla I^{n}(x,y)|}{b}} \mathcal{D}_2 \, I^{n}(x,y) \right)$$

$\mathcal{D}_1$ = operator approximating $\nabla \cdot$

$\mathcal{D}_2$ = operator approximating ∇

} Recall: $\frac{\partial I}{\partial x}(x,y) \approx \underbrace{I(x+1,y) - I(x,y) \text{ etc}}_{\text{linear operator}}$

∴ Starting with $I^{0}(x,y) = I(x,y)$, we iteratively modify $I^{n}(x;y)$.

Such a process is called Anisotropic diffusion image denoising.

How to approximate $\nabla\cdot$ and ∇?

Note: $\nabla\cdot\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$ and $\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$.

We need to approximate $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

In discrete case, f can be considered as $f \in M_{N\times N}$.

Then, we can define $D_x f \in M_{N\times N}$ $\left(\approx \frac{\partial f}{\partial x}\right)$ such that:

$$D_x f(x,y) = f(x+1,y) - f(x,y) \quad \text{for} \quad 0 \leq x, y \leq N-1$$

$$\left(\therefore\ D_x f = h_x * f \text{ for } h_x = \begin{pmatrix} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{pmatrix}\right)$$

Similarly, we can define $D_y f \in M_{N\times N}$ $\left(\approx \frac{\partial f}{\partial y}\right)$ Such that:

$$D_y f(x,y) = f(x,y+1) - f(x,y) \quad \text{for } 0 \leq x, y \leq N-1.$$

$$\left(\therefore\ D_y f = h_y * f \text{ where } h_y = \begin{pmatrix} & 0 & \\ 0 & 1 & 1 \\ & 0 & \end{pmatrix}\right)$$

For our denoising model:

$$\frac{\partial I(x,y;\sigma)}{\partial \sigma} = \nabla\cdot\underbrace{\left(e^{-\frac{|\nabla I(x,y;t)|}{b}}\ \overset{\left(\frac{\partial I}{\partial x},\frac{\partial I}{\partial y}\right)}{\overset{/\!/}{\nabla}} I(x,y;\sigma)\right)}_{\frac{\partial}{\partial x}\left(e^{-|\nabla I(x,y;t)|}\frac{\partial I}{\partial x}(x,y)\right)+\frac{\partial}{\partial y}\left(e^{-|\nabla I(x,y,t)|}\frac{\partial I}{\partial y}(x,y)\right)}$$

It can be discretized as: for $0 \le x,y \le N-1$

$$I^{n+1}(x,y) - I^{n}(x,y) = h_x * I_1^n(x,y) + h_y * I_2^n(x,y)$$ where

$$I_1^n(x,y) = e^{-\frac{|\nabla I^n(x,y)|}{b}} h_x * I(x,y)$$ and

$$I_2^n(x,y) = e^{-\frac{|\nabla I^n(x,y)|}{b}} h_y * I(x,y)$$

$$\left(|\nabla I^n(x,y)| = \sqrt{|h_x * I^n(x,y)|^2 + |h_y * I^n(x,y)|^2}\right)$$

Let $I^0 = I$. Then: I^1 can be obtained by:

$$I^1 = I^0 + h_x * I_1^0 + h_y * I_2^0$$

Successively, given I^1, we get $I^2 \to I^3 \to \dots \to I^n$ until we obtain a desirable result.

Image denoising using energy minimization

Let g be a noisy image corrupted by additive noise n.

Then: $g(x,y) = \underbrace{f(x,y)}_{\text{Clean image}} + \underbrace{n(x,y)}_{\text{noise}}$

Recall: Laplacian masking: $g = f - \Delta f$ (Obtain a $\overset{\text{(non-smooth)}}{\text{sharp}}$ image from a smooth image)

Conversely, to get a smooth image f from a non-smooth image g, we can solve the PDE for f: $\underbrace{-\Delta f + f}_{\text{unknown}} = \underbrace{g}_{\text{known}}$

We will show that solving the above equation is equivalent to minimizing something:

$$E(f) = \iint \Big(f(x,y) - g(x,y)\Big)^2 dx\,dy + \iint \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 dx\,dy$$