Lecture 18:

- **Non-local mean filter**

Let g be a $N \times N$ image.

$X = (x, y)$, $X' = (x', y')$ } Two pixels.

Define: $S_X = \{(x+s, y+t) : -a \leq s, t \leq a\}$; $S_{X'} = \{(x'+s, y'+t) : -a \leq s, t \leq a\}$

Define: $g_X = g|_{S_X}$ and $g_{X'} = g|_{S_{X'}}$.

$\underbrace{(2a+1)}_{m} \times \underbrace{(2a+1)}_{m}$ image

Let $\tilde{g}_X$ = smoothed image of g_X by Gaussian smoothing

$\tilde{g}_{X'}$ = smoothed image of $g_{X'}$ by Gaussian smoothing.

Define the weight: $w(X, X') = e^{\frac{-\|\tilde{g}_X - \tilde{g}_{X'}\|_F^2}{h^2}}$

noise level parameter

(small when X and X' are far away)

far away in term of small images

$$\boxed{\text{Non-local mean filter of } g: \quad \hat{g} = \frac{\sum_{X' \in \text{image domain}} w(X, X')\, g(X')}{\sum_{X' \in \text{image domain}} w(X, X')}}$$

Preliminary

Integration by part:

- $\displaystyle \int_a^b f(x)\, g'(x)\, dx = -\int_a^b f'(x)\, g(x)\, dx + f(x)\, g(x)\Big|_{x=a}^{x=b}$

Now, Suppose $g(a, y) = g(b, y) = 0$. Then:

$$\int_a^b \int_a^b f(x,y)\, \frac{\partial g}{\partial x}(x,y)\, dx\, dy = \int_a^b \left(-\int_a^b \frac{\partial f}{\partial x}(x,y)\, g(x,y)\, dx + f(x,y)\, g(x,y)\Big|_{x=a}^{x=b} \right) dy$$

$$= -\int_a^b \int_a^b \frac{\partial f}{\partial x}(x,y)\, g(x,y)\, dx\, dy$$

Suppose: $g(a, y) = g(b, y) = 0$, $\dfrac{\partial g}{\partial x}(a, y) = \dfrac{\partial g}{\partial y}(b, y) = 0$. Then:

$$\int_a^b \int_a^b f(x,y)\, \frac{\partial^2 g}{\partial x^2}(x,y)\, dx\, dy = \int_a^b \left(-\int_a^b \frac{\partial f}{\partial x}(x,y)\, \frac{\partial g}{\partial x}(x,y)\, dx \right) dy + \int_a^b \cancelto{0}{f(x,y)\, \frac{\partial g}{\partial x}(x,y)\Big|_{x=a}^{x=b}}\, dy$$

$$= \int_a^b \int_a^b \frac{\partial^2 f}{\partial x^2}(x,y)\, g(x,y)\, dx\, dy - \int_a^b \frac{\partial f}{\partial x}(x,y)\, g(x,y)\Big|_{x=a}^{x=b} dy$$

$$= \int_a^b \frac{\partial^2 f}{\partial x^2}(x,y)\, g(x,y)\, dx\, dy$$

Similarly, suppose $g(x,a) = g(x,b) = 0$, $\frac{\partial g}{\partial x}(x,a) = \frac{\partial g}{\partial y}(x,b) = 0$, then:

$$\int_a^b \int_a^b f(x,y) \frac{\partial^2 g}{\partial y}(x,y)\, dx\, dy = \int_a^b \int_a^b \frac{\partial^2 f}{\partial y^2}(x,y)\, g(x,y)\, dx\, dy$$

In summary, if $g(x,y)$ and $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ vanish on the boundary of $[a,b] \times [a,b]$,

then: $$\int_a^b \int_a^b f(x,y)\, \Delta g(x,y)\, dx\, dy = \int_a^b \int_a^b \Delta f(x,y)\, g(x,y)\, dx\, dy$$

Image denoising by solving Anisotropic heat diffussion

Consider the PDE:

$$(*)\quad \frac{\partial I(x,y,t)}{\partial t} = t\left[\frac{\partial^2 I(x,y,t)}{\partial x^2} + \frac{\partial^2 I(x,y,t)}{\partial y^2}\right] = t\,\nabla\cdot(\nabla I) = t\,\Delta I$$

($\nabla\cdot$ = divergence ; $\nabla\cdot(v_1, v_2) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$) $\left(\nabla I = \left(\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y}\right)\right)$

Then: $g(x,y,t) = \frac{1}{2\pi t^2} e^{-(x^2+y^2)/2t^2}$ satisfies $(*)$.

Observation: We'll see that Gaussian filter is approximately solving $(*)$.

Given an image $I(x,y)$ (Assume I is continuously defined on the 2D domain $[a,b]\times[a,b]$)

Gaussian filter = convolution of I with the Gaussian function:

$$\tilde{I}(x,y,t) = I * g(x,y,t) = \int_a^b\int_a^b g(u,v;t)\, I(x-u, y-v)\, du\, dv$$

(Analogous to discrete convolution)

$$\tilde{I}(x,y,0) = I(x,y)$$

$$\therefore \frac{\partial \tilde{I}}{\partial t} = \int_a^b \int_a^b \frac{\partial g(u,v;t)}{\partial t} I(x-u, y-v)\, du\, dv$$

$$= t \int_a^b \int_a^b \frac{\partial^2 g(u,v;t)}{\partial u^2} I(x-u, y-v)\, du\, dv + t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^2 g}{\partial v^2}(u,v;t)\, I(x-u, y-v)\, du\, dv$$

$$= t \int_a^b \int_a^b g(u,v;t) \frac{\partial I}{\partial x^2}(x-u, y-v)\, du\, dv + t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(u,v;t) \frac{\partial I}{\partial y^2}(x-u, y-v)\, du\, dv$$

$$= t \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \int_a^b \int_a^b g(u,v;t)\, I(x-u, y-v)\, du\, dv.$$

$$= t\, \nabla \cdot (\nabla \tilde{I})(x,y,t)$$

$\therefore$ Gaussian filter = solving PDE!!

This motivates us to solve imaging problem by considering PDE.

<u>Remark:</u> Discretize:

$$\frac{\partial I(x,y,t)}{\partial t} = t\left[\frac{\partial^2 I(x,y,t)}{\partial x^2} + \frac{\partial^2 I(x,y,t)}{\partial y^2}\right]$$

gives:

$$\frac{\tilde{I}(x,y,t) - I(x,y)}{t} = t\,\Delta I(x,y)$$

$$\Rightarrow \tilde{I}(x,y,t) = I(x,y) + t^2\,\Delta I(x,y)$$

Note that $\Delta I(x,y)$ is large if (x,y) is on the edge of the object in an image.

Hence, the intensity of $\tilde{I}(x,y,t)$ at (x,y) changes a lot.

Thus, the edge is smoothed out under Gaussian filtering.

Now, we consider a simplified PDE:

$$(*) \quad \frac{\partial I(x,y;t)}{\partial t} = \Delta I(x,y,t).$$

Discretize it: (we consider a sequence of images: $I^0, I^1, I^2, \ldots, I^n, \ldots$) that satisfies $(*)$)

$$\underbrace{I^{n+1}}_{\text{image at time } n+1}(x,y) - \underbrace{I^{n}}_{\text{image at time } n}(x,y) = \Delta I^n(x,y) = p * I^n \qquad p = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\therefore \quad I^{n+1}(x,y) = I^n(x,y) + \Delta I^n(x,y)$$

<u>Remark:</u> PDE gives us a recursive relationship to obtain a sequence of images $\underset{\substack{\shortparallel \\ I}}{I^0}, I^1, \ldots, I^n, \ldots$, which are smoother and smoother.