Lecture 16

Method 4: Constrained least square filtering

Disadvantages of Wiener's filter:

① $|N(u,v)|^2$ and $|F(u,v)|^2$ must be known / guessed

② Constant estimation of ratio is not always suitable

Goal: Consider a least square minimization model.

Let $g = \underset{\substack{\uparrow \\ \text{degradation}}}{h} * f + \underset{\nwarrow \text{ noise}}{n}$

In matrix form, $\vec{g} = D\vec{f} + \vec{n}$, $\quad \vec{g}, \vec{f}, \vec{n} \in \mathbb{R}^{N^2}$, $D \in M_{N^2 \times N^2}$

$\vec{g} = \mathcal{S}(g)$ ← Stacked image of g

$\vec{f} = \mathcal{S}(f)$, $\quad \vec{n} = \mathcal{S}(n)$

D ← transformation matrix of $h * f$ (or f)

Constrained least square problem:

Given $\vec{g}$, we need to find an estimation of $\vec{f}$ such that it minimizes:

$$E(f) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} |\Delta f(x,y)|^2$$ subject to the constraint:

$$\|\vec{g} - D\vec{f}\|^2 = \varepsilon$$

$$\left(\Delta f(x,y) \approx f(x+1,y) + f(x,y+1) + f(x-1,y) + f(x,y-1) - 4f(x,y)\right)$$

Δ is the Laplacian in the discrete case

Let $\overrightarrow{p*f} = S(p*f) = L\vec{f}$ — transformation matrix representing the convolution with p.

Then: $E(\vec{f}) = (L\vec{f})^T(L\vec{f})$

We will prove:

<u>Theorem:</u> The constrained least square problem has the optimal solution in the spatial domain that satisfies:

$$(D^T D + \gamma L^T L)\vec{f} = D^T \vec{g}$$

for some suitable parameter γ.

In the frequency domain,

$$\hat{F}(u,v) := DFT(f)(u,v) = \frac{1}{N^2}\,\frac{\overline{H(u,v)}}{|H(u,v)|^2 + \gamma\,|P(u,v)|^2}\,G(u,v)$$

$\Big(H = DFT(h)$; $G(u,v) = DFT(g)$; $P(u,v) = DFT(p)$ where

$$p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & 1 & -4 & 1 & \vdots \\ \vdots & & 1 & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix} \Big)$$

Remark: Constrained least square filtering:

$$T(u,v) = \frac{1}{N^2} \frac{\overline{H(u,v)}}{|H(u,v)|^2 + \gamma |P(u,v)|^2}$$

Let $\tilde{F}(u,v) = T(u,v)\, G(u,v)$

Compute Inverse DFT of $\tilde{F}(u,v)$.

How about in the frequency (Fourier) domain?

Two important theorems

Notation: Let $\mathcal{O}$ be a linear transformation defined by:

$\mathcal{O}(f) = k * f$ for all $f \in M_{N \times N}(\mathbb{R})$, where

$k \in M_{N \times N}(\mathbb{R})$.

let $D \in M_{N^2 \times N^2}(\mathbb{R})$ be the transformation matrix representing $\mathcal{O}$.

at is, $\mathcal{S}(\mathcal{O}(f)) = D\,\mathcal{S}(f)$. $\in \mathbb{R}^{N^2}$

Here, $\mathcal{S}$ is the stacking operator. $\mathcal{S}(I)$ is the vectorized image of I

(1st col of I becomes first n entries of $\mathcal{S}(I)$, 2nd col of I becomes second n entries of $\mathcal{S}(I)$, ..., etc)

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Theorem 1: Let $\mathbb{K} = DFT(k)$. Then:

$$D = W \Lambda_D W^{-1} \quad \text{and} \quad D^T = W \overline{\Lambda_D} W^{-1} \quad \text{where:}$$

$$\Lambda_D = N^2 \begin{pmatrix} \underbrace{\mathbb{K}(0,0) & & & }_{\text{1st col of } \mathbb{K} \text{ becomes 1st } n \text{ diagonal entries}} & & & & & \\ & \mathbb{K}(1,0) & & & & & & & \\ & & \ddots & & & & & & \\ & & & \mathbb{K}(N-1,0) & & & & & \\ & & & & \underbrace{\mathbb{K}(0,1) & & }_{\text{2nd col of } \mathbb{K} \text{ becomes 2nd diagonal entries}} & & & \\ & & & & & \ddots & & & \\ & & & & & & \mathbb{K}(N-1,1) & & \\ & & & & & & & \ddots & \\ & & & & & & & & \mathbb{K}(0,N-1) \\ & & & & & & & & & \ddots \\ & & & & & & & & & & \mathbb{K}(N-1,N-1) \end{pmatrix}$$

for $W = W_N \otimes W_N$ where $W_N = \left(\frac{1}{\sqrt{N}} e^{\frac{2\pi j}{N} mn} \right)_{0 \le m,n \le N-1} \in M_{N \times N}(\mathbb{C})$

Example: Let $\mathcal{O}(f) = k * f$ where $K = DFT(k) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\underset{M_{2\times2}}{\uparrow}$

Let $D \in M_{4\times4}(\mathbb{R})$ be the transformation matrix of $\mathcal{O}$.

Then: $D = W \Lambda_D W^{-1}$ where $\Lambda_D = \begin{pmatrix} a & & & 0 \\ & c & & \\ & & b & \\ 0 & & & d \end{pmatrix}$

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and $W = W_2 \otimes W_2$ $\left(W_2 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \right)$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} & \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\ \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} & -\frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \end{pmatrix}$$

Theorem 2: Let $W = W_N \otimes W_N \in M_{N^2 \times N^2}$.

Let $f = (f_{mn})_{0 \le m,n \le N-1} \in M_{N \times N}(\mathbb{R})$. Consider $S(f) = \vec{f} \in \mathbb{R}^{N^2}$.

Then:

$$W^{-1}\vec{f} = N \begin{pmatrix} F(0,0) \\ F(1,0) \\ \vdots \\ F(N-1,0) \\ F(0,1) \\ F(1,1) \\ \vdots \\ F(N-1,1) \\ \vdots \\ F(0,N-1) \\ F(1,N-1) \\ \vdots \\ F(N-1,N-1) \end{pmatrix} \begin{array}{l} \left.\begin{matrix} \vphantom{F(0,0)} \\ \vphantom{F(1,0)} \\ \vphantom{\vdots} \\ \vphantom{F(N-1,0)} \end{matrix}\right\} \text{1st col of } F \\ \left.\begin{matrix} \vphantom{F(0,1)} \\ \vphantom{F(1,1)} \\ \vphantom{\vdots} \\ \vphantom{F(N-1,1)} \end{matrix}\right\} \text{2nd col of } F \\ \vphantom{\vdots} \\ \left.\begin{matrix} \vphantom{F(0,N-1)} \\ \vphantom{F(1,N-1)} \\ \vphantom{\vdots} \\ \vphantom{F(N-1,N-1)} \end{matrix}\right\} N^{th} \text{ col of } F \end{array}$$

where $F = DFT(f)$

Example: Assume that:

$$G = \begin{pmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{pmatrix} \quad \text{and} \quad W_3^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & exp\left(-\frac{2\pi j}{3}\right) & exp\left(-\frac{2\pi j}{3}2\right) \\ 1 & exp\left(-\frac{2\pi j}{3}2\right) & exp\left(-\frac{2\pi j}{3}\right) \end{pmatrix}$$

Then:

$$W^{-1} = W_3^{-1} \otimes W_3^{-1} = \frac{1}{3} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}4} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}3} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2}
\end{pmatrix}$$

$$W^{-1}\vec{g} = \frac{1}{3}\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2}
\end{pmatrix}
\begin{pmatrix} g_{00} \\ g_{10} \\ g_{20} \\ g_{01} \\ g_{11} \\ g_{21} \\ g_{02} \\ g_{12} \\ g_{22} \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}
g_{00} + g_{10} + g_{20} + g_{01} + g_{11} + g_{21} + g_{02} + g_{12} + g_{22} \\
g_{00} + g_{10}e^{-\frac{2\pi j}{3}} + g_{20}e^{-\frac{2\pi j}{3}2} + g_{01} + g_{11}e^{-\frac{2\pi j}{3}} + g_{21}e^{-\frac{2\pi j}{3}2} + g_{02} + g_{12}e^{-\frac{2\pi j}{3}} + g_{22}e^{-\frac{2\pi j}{3}2} \\
\vdots \\
\vdots \\
\vdots
\end{pmatrix}
\begin{matrix} = 3^2 G(0,0) \\ = 3^2 G(1,0) \\ \vdots \\ \\ \\ \end{matrix}$$

$G = \mathrm{DFT}(g)$

$\therefore \quad W^{-1}\vec{g} = 3\,\mathcal{S}(G)$

Suppose D is the transformation matrix representing the convolution with h.

(In other words, if $g = h * f$, then: $\underset{\in \mathbb{R}^{N^2}}{\vec{g}} = \underset{\in M_{N^2 \times N^2}}{D} \vec{f} \in \mathbb{R}^{N^2}$)

Let $H = \mathrm{DFT}(h) \in M_{N \times N}$

Diagonalization of D:

$$D^T D = N^4 W \left[\begin{pmatrix} |H(0,0)|^2 & & & & & & & \\ & |H(1,0)|^2 & & & & & & \\ & & \ddots & & & & & \\ & & & |H(N-1,0)|^2 & & & & \\ & & & & |H(0,1)|^2 & & & \\ & & & & & \ddots & & \\ & & & & & & |H(N-1,1)|^2 & \\ & & & & & & & \ddots \\ & & & & & & & & |H(0,N-1)|^2 \\ & & & & & & & & & \ddots \\ & & & & & & & & & & |H(N-1,N-1)|^2 \end{pmatrix} \right] W^{-1}$$

Stack H to form the diagonal matrix.

Suppose L is the transformation matrix representing the convolution with p.

(In other words, if $g = p * f$, then: $\underset{\mathbb{R}^{N^2}}{\vec{g}} = \underset{M_{N^2 \times N^2}}{L} \vec{f} \in \mathbb{R}^{N^2}$)

Let $P = \mathrm{DFT}(p) \in M_{N \times N}$

Diagonalization of L:

$$L^T L = N^4 W \Bigg[\underbrace{\begin{pmatrix} |P(0,0)|^2 & & & & & & & \\ & |P(1,0)|^2 & & & & & & \\ & & \ddots & & & & & \\ & & & |P(N-1,0)|^2 & & & & \\ & & & & |P(0,1)|^2 & & & \\ & & & & & \ddots & & \\ & & & & & & |P(N-1,1)|^2 & & \\ & & & & & & & \ddots & \\ & & & & & & & & |P(0,N-1)|^2 & \\ & & & & & & & & & \ddots \\ & & & & & & & & & & |P(N-1,N-1)|^2 \end{pmatrix}}_{\text{Stack } P \text{ to form the diagonal matrix.}} \Bigg] W^{-1}$$

Combining these information and substitute into the "governing" equation :

$$(D^TD + \gamma L^TL)\vec{f} = D^T\vec{g},$$

We get: $\cancel{W}(\Lambda_D^*\Lambda_D + \gamma\Lambda_L^*\Lambda_L)W^{-1}\vec{f} = \cancel{W}\Lambda_D^* W^{-1}\vec{g}$

We can check that:

①
$$\boxed{\Lambda_D^*\Lambda_D = \begin{pmatrix} N^4|H(0,0)|^2 & & & & \\ & N^4|H(1,0)|^2 & & & \\ & & \ddots & & \\ & & & N^4|H(N-1,0)|^2 & \\ & & & & \ddots & \\ & & & & & N^4|H(N-1,N-1)|^2 \end{pmatrix}}$$

②
$$\boxed{\Lambda_L^*\Lambda_L = \begin{pmatrix} N^4|P(0,0)|^2 & & & & \\ & N^4|P(1,0)|^2 & & & \\ & & \ddots & & \\ & & & N^4|P(N-1,0)|^2 & \\ & & & & \ddots & \\ & & & & & N^4|P(N-1,N-1)|^2 \end{pmatrix}}$$

③ $\boxed{W^{-1}\vec{f} = N\mathcal{S}(F), W^{-1}\vec{g} = N\mathcal{S}(G)}$ where $F = DFT(f), G = DFT(g)$.

Combining these information and substitute into the "governing" equation :

$$(D^T D + \gamma L^T L)\vec{f} = D^T \vec{g} ,$$

We get: $\cancel{W}(\Lambda_D^* \Lambda_D + \gamma \Lambda_L^* \Lambda_L) W^{-1}\vec{f} = \cancel{W}\Lambda_D^* W^{-1}\vec{g}$

$$\therefore \quad N^4 \begin{pmatrix} |H(0,0)|^2+\gamma|P(0,0)|^2 & & & & \\ & |H(1,0)|^2+\gamma|P(1,0)|^2 & & & \\ & & \ddots & & \\ & & & |H(N-1,0)|^2+\gamma P(N-1,0)|^2 & \\ & & & & \ddots \end{pmatrix} N \begin{pmatrix} F(0,0) \\ F(1,0) \\ \vdots \\ F(N-1,0) \\ \vdots \end{pmatrix} =$$

$$N^2 \begin{pmatrix} \overline{H(0,0)} & & & & \\ & \overline{H(1,0)} & & & \\ & & \ddots & & \\ & & & \overline{H(N-1,0)} & \\ & & & & \ddots \end{pmatrix} N \begin{pmatrix} G(0,0) \\ G(1,0) \\ \vdots \\ G(N-1,0) \\ \vdots \end{pmatrix}$$

Combining all these, we get for every (u,v),

$$N^4[|H(u,v)|^2 + \gamma|P(u,v)|^2]NF(u,v) = N^2\overline{H(u,v)}NG(u,v)$$

$$\Rightarrow \boxed{N^2\frac{|H(u,v)|^2 + \gamma|P(u,v)|^2}{\overline{H(u,v)}}F(u,v) = G(u,v)}$$

Summary: Constrained least square filtering minimizes:

$$E(\vec{f}) = (L\vec{f})^T(L\vec{f})$$

subject to the constraint that:

$$\|\underbrace{\vec{g} - H\vec{f}}_{\vec{n}}\|^2 = \varepsilon$$

(allow fixed amount of noise)

Image sharpening in the frequency domain

Goal: Enhance image so that it shows more obvious edges.

Method 1: Laplacian masking

Recall that : $\Delta f(x,y) = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$.

In the discrete case, $\Delta f(x,y) \approx f(x+1,y) + f(x,y+1) + f(x,y-1) + f(x-1,y) - 4f(x,y)$

or $\Delta f \approx p * f$ where $p = \begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix}$

We can observe that $-\Delta f$ captures the edges of the image (leaving other region zero)

$\therefore$ Shapen image $= f + \overbrace{(-\Delta f)}^{\text{add more edges}}$

In the frequency domain: $DFT(g) = DFT(f) - DFT(\overset{\overset{p*f}{\parallel}}{\Delta f})$

$= DFT(f) - c\,DFT(p).DFT(f)$

$\therefore DFT(g) = [1 - \underset{\underset{c\,DFT(p)}{\parallel}}{H_{laplacian}(u,v)}]\, DFT(f)(u,v)$

Method 2: Unsharp masking

Idea: Add high-frequency component

Definition: Let f = input image (blurry)

Let f_{smooth} = smoother image

Define a sharper image as:

$$g(x,y) = f(x,y) + k\left(f(x,y) - f_{smooth}(x,y)\right)$$

When $k=1$, the method is called unsharp masking.

When $k>1$, the method is called highboost filtering.

In the frequency domain, let $DFT(f_{smooth})(u,v) = \underbrace{H_{LP}}_{\text{Low-pass filter}}(u,v)\, DFT(f)(u,v)$

Then: $DFT(g) = [1 + k(1 - H_{LP}(u,v)]\, DFT(f)(u,v)$

Image denoising in the spatial domain

Definition: Linear filter = modify pixel value by a linear combination of pixel values of local neighbourhood.

Example 1: Let f be an $N \times N$ image. Extend the image periodically. Modify f to $\tilde{f}$ by:

$$\tilde{f}(x,y) = f(x,y) + 3f(x-1,y) + 2f(x+1,y).$$

This is a linear filter.

Example 2: Define

$$\tilde{f}(x,y) = \frac{1}{4}\left(f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)\right)$$

This is also a linear filter.

<u>Recall</u>: The discrete convolution is defined as:

$$I * H(u,v) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} I(u-m, v-n) H(m,n)$$

(Linear combination of pixel values around (u,v))

Therefore, **Linear filter is equivalent to a discrete convolution**.

Geometric illustration

$$\begin{tikzcd}
I \arrow[d, "* H"] \arrow[dr, phantom, "\cong"] & \hat{I} \arrow[d, "\times \hat{H}"] \\
\tilde{I} \arrow[r, shift left, "DFT"] & \hat{\tilde{I}} \arrow[l, shift left, "iDFT"]
\end{tikzcd}$$

(spatial) (frequency)

Commonly used filter (linear)

- **Mean filter:**

$$H = \frac{1}{9} \begin{array}{cc} & \begin{array}{ccc} -1 & 0 & 1 \end{array} \\ & \left(\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right) \begin{array}{c} -1 \\ 0 \\ 1 \end{array} \end{array}$$

(Here, we only write down the entries of the matrix for indices $-1 \leq k, l \leq 1$ for simplicity. All other matrix entries are equal to 0.)

This is called the *mean filtering with window size* 3×3.

- **Gaussian filter:** The entries of H are given by the Gaussian function $g(r) = exp\left(-\frac{r^2}{2\sigma^2}\right)$, where $r = \sqrt{x^2 + y^2}$.