# Lecture 12:

**Recap:**

- Given an image $I$, $\hat{I} = DFT(I)$ can be computed:

$$\hat{I} = U I U \quad \text{where} \quad U = (U_{kl})_{0 \le k,l \le N-1}$$

$$U_{kl} = \frac{1}{N} e^{-j\frac{2\pi}{N}(kl)}$$

- Given $\hat{I}$, $I$ can be constructed:

$$I = U^{-1} \hat{I} U^{-1} = (N U^{*}) \hat{I} (N U^{*})$$

$$\left( U^{*} = (\overline{U})^{T} \right)$$

- The m row n col entry of $\hat{I}$ is given by:

$$\hat{I}(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} I(k,l)\, e^{-j\frac{2\pi}{N}(km+ln)}$$

- The m row n col entry of $I$ :

$$I(m,n) = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \hat{I}(k,l)\, e^{j\frac{2\pi}{N}(km+ln)}$$

Image enhancement in the frequency domain (by modifying Fourier coefficients)

Goal: 1. Remove $\underbrace{\text{high-frequency components}}_{\text{noise}}$ (low-pass filter) for image denoising.

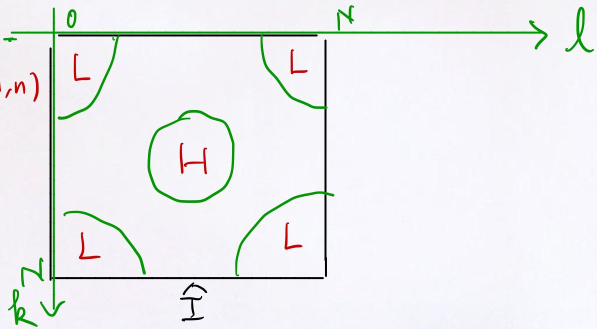
2. Remove $\underbrace{\text{low-frequency components}}_{\text{non-edge}}$ (high-pass filter) for the extraction of image details.

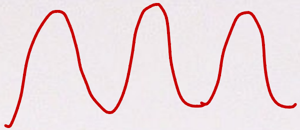
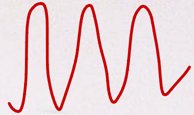
If an image $I$ takes indices between $0$ to $N$, then:

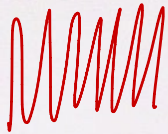
$$I(m,n) = \sum_{k=0}^{N} \sum_{l=0}^{N} \underbrace{\hat{I}(k,l)}_{\text{DFT}(I)} e^{j\frac{2\pi}{N}(km+ln)} \qquad \text{where } 0 \le m,n \le N$$

$$I(m,n) = \sum_{k} \sum_{l} \hat{I}(k,l)\, g_{kl}(m,n)$$

$$g_{kl}(m,n) = e^{j\frac{2\pi}{N}(km+ln)}$$



 = a  + b 

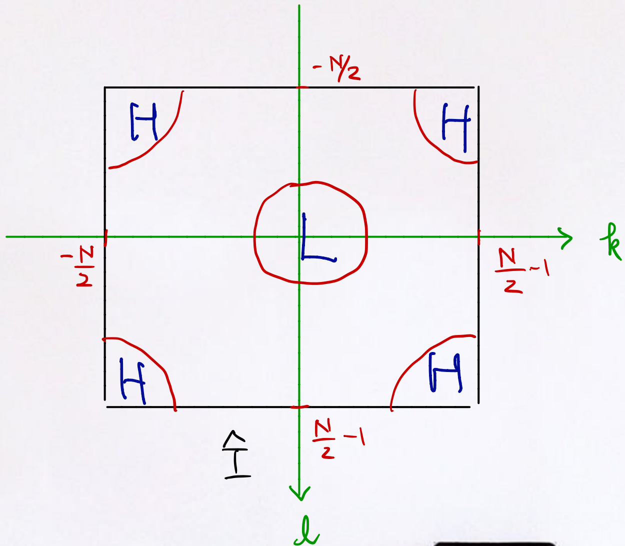
+ c 

To remove noise, truncate c (let $c = 0$)

If an image $I$ takes indices between $-\frac{N}{2}$ to $\frac{N}{2}-1$; then:

$$I(m,n) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1} \underbrace{\hat{I}(k,l)}_{DFT(I)} \, e^{j\frac{2\pi}{N}(km+ln)}$$

where $-\frac{N}{2} \le m,n \le \frac{N}{2}-1$




(Centralization)

## Procedures for image processing by modifying Fourier coefficients

Given an image $I = (I_{ij})_{-\frac{N}{2} \le i,j \le \frac{N}{2}-1}$

Compute DFT of $I$ ( Denote $\hat{I} = \mathrm{DFT}(I)$ )

Then: obtain a new DFT matrix, $\hat{I}^{new}$, by:

$$\hat{I}^{new} = H \odot \hat{I}$$

( Here $H \odot \hat{I}(u,v) = H(u,v)\hat{I}(u,v)$ )

pixel-wise multiplication

H is a suitable filter.

Finally, obtain an improved image by inverse DFT:

$$I^{new} = \underbrace{i\mathrm{DFT}}_{\text{inverse DFT}}(\hat{I}^{new})$$

Note: Let $h = \underbrace{\mathrm{iDFT}}_{\text{inverse DFT}}(H)$

$$I^{new} = C\, h * I$$

$$H \odot \hat{I} \xrightarrow{\text{inverse DFT}} \underset{\substack{\uparrow \\ \text{normalizing constant}}}{C}\; h * I$$

$$I^{new}(x,y) = h * I(x,y) = C \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1} h(x-k,\, y-l)\, I(k,l)$$

If $h(\tilde{x}, \tilde{y})$ is non-zero only when $\tilde{x} \approx 0$ and $\tilde{y} \approx 0$, then the above summation has only a few non-zero terms (i.e. terms for $k$ close to $x$ and $l$ close to $y$)

If not, every $I(k,l)$ can affect $I^{new}(x,y)$.

## Recap: <u>Example of Low-pass filters for image denoising</u>

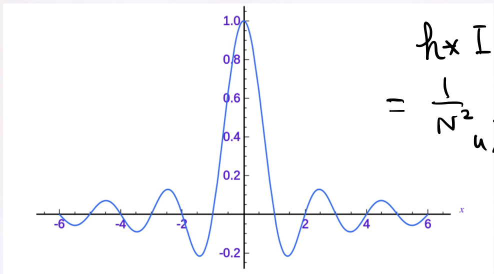
Assume that we work on the centered spectrum!

That is, consider $\hat{F}(u,v)$ where $-N/2 \le u \le N/2 - 1$, $-N/2 \le v \le N/2 - 1$.

1 Ideal low pass filter (ILPF):

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) := u^2 + v^2 \le D_0^2 \\ 0 & \text{if } D(u,v) > D_0^2 \end{cases}$$

In 1-dim cross-section, $\mathcal{F}^{-1}(H(u,v))$ looks like:




$$h * I(x,y) = \frac{1}{N^2} \sum_{u,v} h(x-u, y-v)\, I(u,v)$$

every pixel values of $I$ has an effect on $h * I(x,y)$ !!

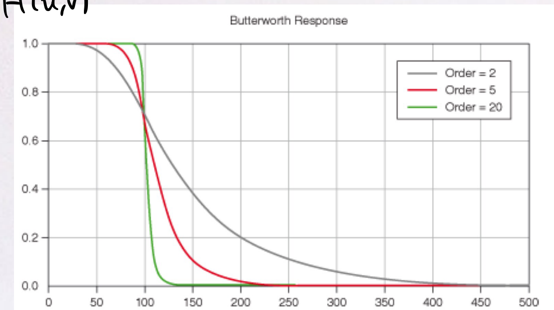
Good: Simple

Bad : Produce ringing effect!

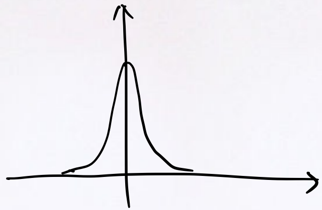
2. Butterworth low-pass filter (BLPF) of order $n$ ( $n \geq 1$ integer) :

$$H(u,v) = \frac{1}{1 + \left(D(u,v)/D_0\right)^n}$$

$H(u,v)$ in 1-dim




$\mathcal{F}^{-1}(H(u,v))$ in 1-dim

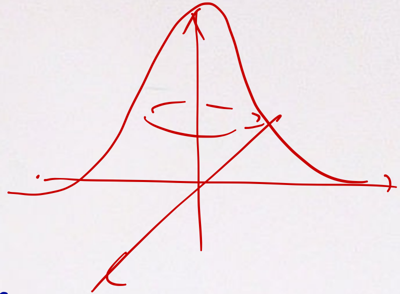


Good: Produce less / no visible ringing effect if $n$ is carefully chosen!!

3. Gaussian low-pass filter

$$H(u,v) = \exp\left(-\frac{\overset{u^2+v^2}{D(u,v)}}{2\sigma^2}\right)$$

$\sigma$ = spread of the Gaussian function



F.T. of Gaussian is also Gaussian!!

Good: No visible ringing effect!!

# Examples for high-pass filtering for feature extraction

1. Ideal high-pass filter: (IHPF)

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0^2 \\ 1 & \text{if } D(u,v) > D_0^2 \end{cases}$$

Bad: Produce ringing

2. Butterworth high-pass filter:

$$H(u,v) = \frac{1}{1 + \left(\frac{D_0}{D(u,v)}\right)^n}$$

$(H(u,v) = 0 \text{ if } D(u,v) = 0)$

Choose the right $n$

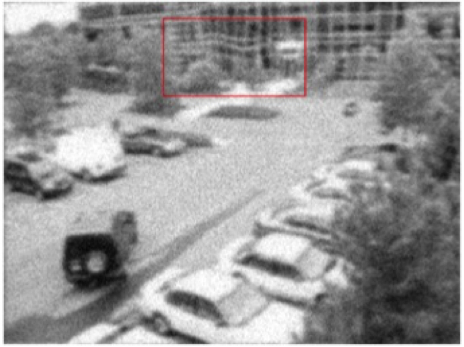
Good: Less ringing

3. Gaussian high-pass filter

$$H(u,v) = 1 - e^{-\left(\frac{D(u,v)}{2\sigma^2}\right)}$$

Good: No visible ringing!

# Image deblurring



Atmospheric turbulence

Motion Blur

Speeding problem

## Image deblurring in the frequency domain:

## Mathematical formulation of image blurring

$$f \xrightarrow{D} g$$

(not a matrix, just a transformation)

$f$: clean, original image

$g$: blurry image of $f$

Let $g$ be the observed (blurry) image.

Let $f$ be the original (good) image.

Model $g$ as: $g = D(f) + n$

where $D$ is the degradation function/operator and $n$ is the additive noise.

Assumption on $D$:

1. $D$ is position invariant:

   Let $g(x,y) = D(f)(x,y)$ and let $\tilde{f}(x,y) := f(x-\alpha, y-\beta)$.

   Then: $D(\tilde{f})(x,y) = g(x-\alpha, y-\beta) = D(f)(x-\alpha, y-\beta)$

2. Linear: $D(f_1 + f_2) = D(f_1) + D(f_2)$

   $D(\alpha f) = \alpha D(f)$ where $\alpha$ is a scalar multiplication.

Claim: With the above assumption, we can show that: (assume indices taken between $-\frac{N}{2}$ to $\frac{N}{2}-1$)

$$D(f) = f * h \qquad \text{where}$$

$$h = D(\underbrace{\delta}_{N \times N \text{ matrix}}) \qquad \delta(x,y) = \begin{cases} 1 & \text{if } (x,y)=(0,0) \\ 0 & \text{otherwise} \end{cases}$$

(Next time)