Lecture 11:

Recall:

DFT of a rotated image

Consider a $N \times N$ image g.

Consider a rotated image $\tilde{g}(r,\theta) = g(r, \theta+\theta_0)$ where θ is defined between $-\theta_0$ to $\pi/2 - \theta_0$.

$\therefore$ image g is rotated clockwise by θ_0.

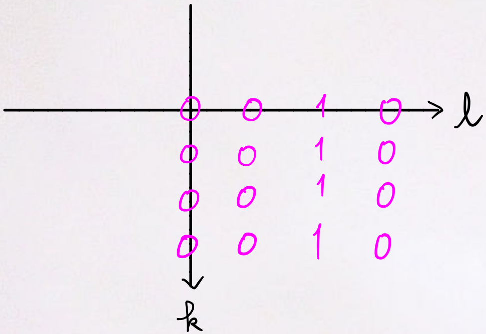
Then:

$$\underbrace{\hat{\tilde{g}}}_{=\,\mathrm{DFT}(\tilde{g})}(\omega,\phi) = \underbrace{\hat{g}}_{=\,\mathrm{DFT}(g)}(\omega, \phi+\theta_0).$$

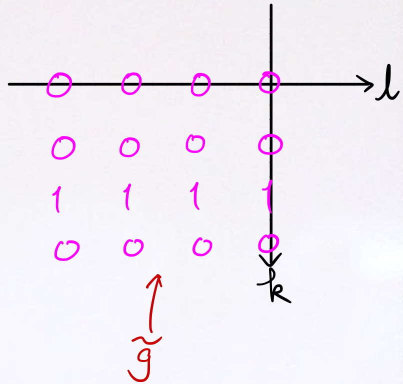
(ϕ is also defined between $-\theta_0$ to $\pi/2 - \theta_0$)

Example: Let $g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Then: $\hat{g} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Note that g in the coordinate system:



Rotated by 90° clockwisely



Note that indices of $\tilde{g}$ are taken as: $\begin{cases} -3 \le l \le 0 \\ 0 \le k \le 3. \end{cases}$

Now, DFT of $\tilde{g}$ = $\hat{\tilde{g}}$ (given by: $\displaystyle\sum_{k=0}^{3}\sum_{l=-3}^{0}\tilde{g}(k,l)\, e^{-j2\pi\left(\frac{km+ln}{4}\right)}$

$$= \begin{array}{c|c} \begin{pmatrix} 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & -1/4 \\ 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & -1/4 \end{pmatrix} & \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} \; k \\ \begin{matrix} l \quad -3 & -2 & -1 & 0 \end{matrix} & \end{array}$$

$\therefore$ $0 \le k \le 3$

$-3 \le l \le 0$

4. DFT of a shifted image

Let $g = (g(k', l'))$ be a $N \times N$ image, where the indices are taken as:

$-k_0 \le k' \le N-1-k_0$ and $-l_0 \le l' \le N-1-l_0$

Let $\tilde{g}$ be shifted image of g defined as:

$\tilde{g}(k,l) = g(k-k_0,\, l-l_0)$ where $0 \le k \le N-1$

$0 \le l \le N-1$

Then: $$\hat{\tilde{g}}(m,n) = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k-k_0,\, l-l_0)\, e^{-j2\pi\left(\frac{km+ln}{N}\right)}$$

$$= \underbrace{\frac{1}{N^2}\sum_{k'=-k_0}^{N-1-k_0}\sum_{l'=-l_0}^{N-1-l_0} g(k', l')\, e^{-j2\pi\left(\frac{k'm+l'n}{N}\right)}}_{\hat{g}(m,n)}\, e^{-j2\pi\left(\frac{k_0 m+l_0 n}{N}\right)}$$

$$\therefore \quad \hat{\hat{g}}(m,n) = \hat{g}(m,n)\, e^{-j2\pi\left(\frac{k_0 m + l_0 n}{N}\right)}$$

<u>Remark:</u> $\hat{g}(m-m_0, n-n_0) = \mathrm{DFT}\left(g \times e^{j2\pi\left(\frac{m_0 k + n_0 l}{N}\right)}\right)$ with carefully chosen indices!

Note:

(Spatial domain) $I * g$ (Linear filtering: Linear combination of neighborhood pixel values)

$$\Big\downarrow \text{DFT}$$

(Frequency domain) $MN\, \underbrace{\hat{I} \odot \hat{g}}_{\text{pixel-wise multiplication}}$ (Modifying the Fourier coefficients by multiplication)

Image enhancement in the frequency domain:

Goal: 1. Remove $\underbrace{\text{high-frequency components}}_{\text{noise}}$ (low-pass filter) for image denoising.

2. Remove $\underbrace{\text{low-frequency components}}_{\text{non-edge}}$ (high-pass filter) for the extraction of image details.

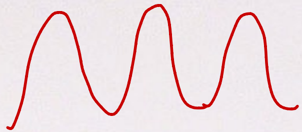
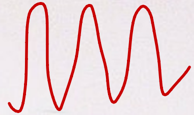
Let $\hat{F}$ be the DFT of an $N \times N$ image F. (indices taken from 0 to $N-1$)

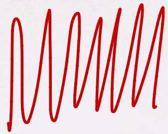
Then: for all $0 \le m, n \le N-1$,

$$\hat{F}(m,n) = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \hat{F}(k,l)\, e^{j\frac{2\pi}{N}(km+ln)}$$

$\therefore$ $\hat{F}(k,l)$ is associated to the complex function $g(m,n) = e^{j\frac{2\pi}{N}(km+ln)}$

Goal: Remove "jumpy" components by setting suitable $\hat{F}(k,l)$ to zero.

 $= a$  $+ \, b$ 

$+ \, c$ 

To remove noise, truncate c (let $c = 0$)

Observation:

1. When k and l are close to 0, $\hat{F}(k,l)$ is associated to $g(m,n) = e^{j\frac{2\pi}{N}(km+ln)}$

$$\approx e^{j\frac{2\pi}{N}(0m+0n)}$$

$$\approx 1 \text{ (constant)}$$

(Not "jumpy")

$\therefore$ Fourier coefficients at the bottom left are associated to low frequency components!

2. When k and l are close to N, $\hat{F}(k,l)$ is associated to

$$g(m,n) = e^{j\frac{2\pi}{N}(km+ln)} \approx e^{j\frac{2\pi}{N}(Nm+Nn)} = e^{j2\pi(m+n)} \approx 1 \quad \text{(Not "jumpy")}$$

$\therefore$ Fourier coefficients at the bottom riget are associated to low frequency components!

2. Similarly, we can check that Fourier coefficients at the 4 corners are associated to low frequency components.

3. Fourier coefficients in the middle are associated to high frequency components:

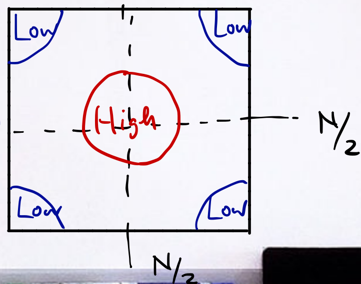
When k and l are close to $N/2$

$\hat{F}(k,l)$ is associated to:

$$g(m,n) = e^{j\frac{2\pi}{N}(km+ln)} \approx e^{j\frac{2\pi}{N}(\frac{N}{2}m+\frac{N}{2}n)}$$

$$= e^{j\pi(m+n)} = (-1)^{m+n}$$

(most "jumpy")



$\therefore$ High-pass filtering

"

Remove coefficients at 4 corners

Low-pass filtering

"

Remove coefficients at the center

Image enhancement in the frequency domain:

Goal: 1. Remove $\underbrace{\text{high-frequency components}}_{\text{noise}}$ (low-pass filter) for image denoising.

2. Remove $\underbrace{\text{low-frequency components}}_{\text{non-edge}}$ (high-pass filter) for the extraction of image details.

High/Low frequency components of $\hat{F}$

Let F be a $N \times N$ image, N = even. Let $\hat{F}$ = DFT of F.

$$\therefore \quad \hat{F}(k, l) = \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} F(m,n)\, e^{-j 2\pi \cdot \left(\frac{mk + nl}{N}\right)}$$

Fourier coefficients of F at (k, l)

Observe that: for $0 \leq k, l \leq \frac{N}{2} - 1$

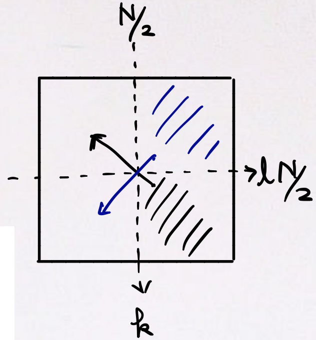
$$\hat{F}\left(\frac{N}{2} + k, \frac{N}{2} + l\right) = \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} F(m,n)\, e^{-j \frac{2\pi}{N}\left(m\left(\frac{N}{2} + k\right) + n\left(\frac{N}{2} + l\right)\right)}$$

$$= \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} F(m,n) (-1)^{m+n} e^{-j \frac{2\pi}{N}(m(-k) + n(-l))}$$

$$= \overline{\frac{1}{N^2}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} F(m,n)\, e^{-j\frac{2\pi}{N}\left(m\left(\frac{N}{2}-k\right)+n\left(\frac{N}{2}-l\right)\right)}}$$

$$= \overline{\hat{F}\left(\frac{N}{2}-k,\ \frac{N}{2}-l\right)}$$

$\therefore$ Computing part of $\hat{F}$ can determine the rest !!



We have:

$$
\begin{aligned}
F(m,n) = & \sum_{0\le k,l\le \frac{N}{2}-1}\left[\hat{F}\left(\frac{N}{2}+k,\frac{N}{2}+l\right)e^{j\frac{2\pi}{N}[(\underline{\frac{N}{2}+k})m+(\underline{\frac{N}{2}+l})n]}\right] \\
& + \sum_{1\le k,l\le \frac{N}{2}-1}\left[\overline{\hat{F}\left(\frac{N}{2}+k,\frac{N}{2}+l\right)}e^{j\frac{2\pi}{N}[(\underline{\frac{N}{2}-k})m+(\underline{\frac{N}{2}-l})n]}\right] \\
& + \sum_{0\le k,l\le \frac{N}{2}-1}\left[\hat{F}\left(\frac{N}{2}+k,\frac{N}{2}-l\right)e^{j\frac{2\pi}{N}[(\underline{\frac{N}{2}+k})m+(\underline{\frac{N}{2}-l})n]}\right] \\
& + \sum_{1\le k,l\le \frac{N}{2}-1}\left[\overline{\hat{F}\left(\frac{N}{2}+k,\frac{N}{2}-l\right)}e^{j\frac{2\pi}{N}[(\underline{\frac{N}{2}-k})m+(\underline{\frac{N}{2}+l})n]}\right] \\
& + \sum_{0\le l\le \frac{N}{2}-1}\hat{F}\left(0,\frac{N}{2}+l\right)e^{j\frac{2\pi}{N}[(\frac{N}{2}+l)n]} + \sum_{1\le l\le \frac{N}{2}-1}\overline{\hat{F}\left(0,\frac{N}{2}+l\right)}e^{j\frac{2\pi}{N}[(\frac{N}{2}-l)n]} \\
& + \sum_{0\le k\le \frac{N}{2}-1}\hat{F}\left(\frac{N}{2}+k,0\right)e^{j\frac{2\pi}{N}[(\frac{N}{2}+k)m]} + \sum_{1\le k\le \frac{N}{2}-1}\overline{\hat{F}\left(\frac{N}{2}+k,0\right)}e^{j\frac{2\pi}{N}[(\frac{N}{2}-k)m]} + \hat{F}(0,0)
\end{aligned}
$$

Centralisation:

Assume periodic conditions on F.

We can let $\tilde{F}(u,v) = \hat{F}(u-\frac{N}{2}, v-\frac{N}{2})$ where $0 \le u \le N-1$
$0 \le v \le N-1$

Then, High-frequency components are located at 4 corners of $\tilde{F}(u,v)$

Low-frequency components are located at center of $\tilde{F}(u,v)$

Let F be an image whose indices are taken between $-\frac{N}{2}$ to $\frac{N}{2}$

Then, DFT(F) is a matrix whose indices are also taken between $-\frac{N}{2}$ to $\frac{N}{2}$.

In this case, Fourier coefficients located at 4 corners of DFT(F) are associated to high-frequency components (jumpy)

Fourier coefficients located in the middle of DFT(F) are associated to low-frequency components (less jumpy)

Procedures for image processing by modifying Fourier coefficients

Given an image $I = (I_{ij})_{-\frac{N}{2} \le i,j \le \frac{N}{2}}$.

Compute DFT of I (Denote $\hat{I} = DFT(I)$)

Then: obtain a new DFT matrix, $\hat{I}^{new}$, by:

$$\hat{I}^{new} = H \odot \hat{I}$$

(Here $H \odot \hat{I}(u,v) = H(u,v)\hat{I}(u,v)$)

$\odot$: pixel-wise multiplication

H is a suitable filter.

Finally, obtain an improved image by inverse DFT:

$$I^{new} = \underbrace{iDFT}_{\text{inverse DFT}}(\hat{I}^{new})$$

Note: Let $h = \underbrace{iDFT}_{\text{inverse DFT}}(H)$

$$H \odot \hat{I} \xrightarrow{\text{inverse DFT}} C \, h * I$$

normalizing constant

Example of Low-pass filters for image denoising

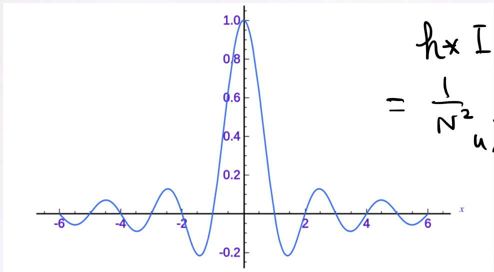
Assume that we work on the centered spectrum!

That is, consider $\hat{F}(u,v)$ where $-N/2 \le u \le N/2 - 1$, $-N/2 \le v \le N/2 - 1$.

1 Ideal low pass filter (ILPF):

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) := u^2 + v^2 \le D_0^2 \\ 0 & \text{if } D(u,v) > D_0^2 \end{cases}$$

In 1-dim cross-section, iDFT(H(u,v)) looks like:



$$h * I(x,y)$$

$$= \frac{1}{N^2} \sum_{u,v} h(x-u, y-v)\, I(u,v)$$

every pixel values of I has an effect on $h * I(x,y)$!!

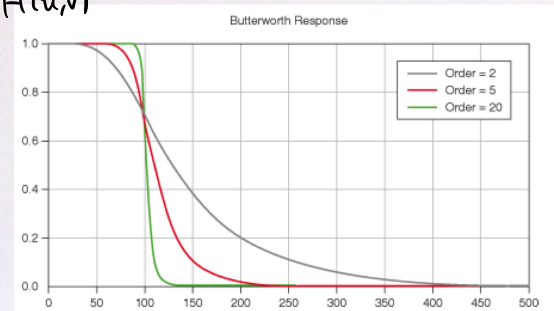
Good: Simple

Bad : Produce ringing effect!

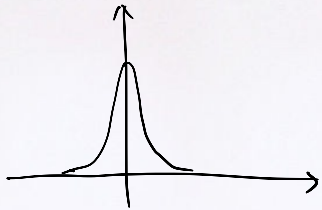
2. Butterworth low-pass filter (BLPF) of order n ($n \geq 1$ integer):

$$H(u,v) = \frac{1}{1 + \left(D(u,v)/D_0\right)^n}$$

$H(u,v)$ in 1-dim



$\mathcal{F}^{-1}(H(u,v))$ in 1-dim

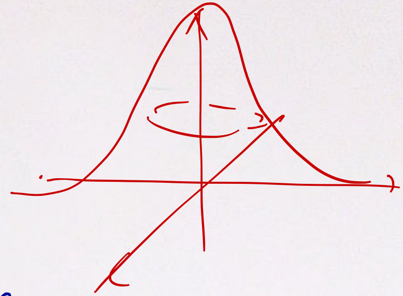


Good: Produce less / no visible ringing effect if n is carefully chosen!!

3. Gaussian low-pass filter

$$H(u,v) = \exp\left(-\frac{\overset{u^2+v^2}{D(u,v)}}{2\sigma^2}\right)$$

σ = spread of the Gaussian function



F.T. of Gaussian is also Gaussian!!

Good: No visible ringing effect!!