# Lecture 10:

## Why is DFT useful in imaging:

1. DFT of convolution:

Recall: $g * w(n,m) = \sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} g(n-n', m-m')\, w(n', m')$

$\left( g, m \in M_{N \times M}(\mathbb{R}) \right)$

Then, the DFT of $g * w = MN\, \mathrm{DFT}(g)\, \mathrm{DFT}(w)$

$\therefore$ DFT of convolution can be reduced to simple multiplication!

Proof:

DFT of $g*w$ at $(p,q)$

$$= \frac{1}{NM}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} g*w(n,m)\, e^{-j2\pi\left(\frac{pn}{N}+\frac{qm}{M}\right)}$$

$$= \frac{1}{NM}\sum_{n'=0}^{N-1}\sum_{m'=0}^{M-1}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} g(n-n', m-m')\, w(n',m')\, e^{-j2\pi\left(\frac{pn}{N}+\frac{qm}{M}\right)}$$

Change of variables:

$n \to n'' = n-n'$

$m \to m'' = m-m'$

$$= \underbrace{\frac{1}{NM}\sum_{n'=0}^{N-1}\sum_{m'=0}^{M-1} w(n',m')\, e^{-j2\pi\left(\frac{pn'}{N}+\frac{qm'}{M}\right)}}_{\hat{w}(p,q)} \underbrace{\sum_{n''=-n'}^{N-1-n'}\sum_{m''=-m'}^{M-1-m'} g(n'',m'')\, e^{-j2\pi\left(\frac{pn''}{N}+\frac{qm''}{M}\right)}}_{T(p,q)}$$

Note that : $g$ and $w$ are periodically extended.

$\therefore\ g(n-N, m) = g(n,m)$ and $g(n, m-M) = g(n,m)$

$$\therefore\ T \equiv \sum_{m''=-m'}^{M-1-m'} e^{-j2\pi\frac{qm''}{M}} \sum_{n''=-n'}^{-1} g(n'',m'')\, e^{-j2\pi\frac{pn''}{N}} + \sum_{m''=-m'}^{M-1-m'} e^{-j2\pi\frac{qm''}{M}} \sum_{n''=0}^{N-1-n'} g(n'',m'')\, e^{-j2\pi\left(\frac{pn''}{N}\right)}$$

Consider $\sum_{n''=-n'}^{-1} g(n'', m'') e^{-j2\pi \frac{pn''}{N}} \overset{n'''=N+n''}{=\!=} \sum_{n'''=N-n'}^{N-1} \underbrace{g(n'''-N, m'')}_{g(n''', m'')} e^{-j2\pi\left(\frac{pn''}{N}\right)} \cancelto{1}{e^{j2\pi p}}$

We can do similar thing for index $m''$.

$$\therefore T = \sum_{m''=0}^{M-1} \sum_{n''=0}^{N-1} g(n'', m'') e^{-j2\pi\left(\frac{pn''}{N} + \frac{qm''}{M}\right)} = MN\,\hat{g}(p, q)$$

$$\therefore \widehat{g * w}(p, q) = MN\,\hat{g}(p, q)\,\hat{w}(p, q)$$

Remark: Conversely, if $x(n, m) = g(n, m)\,w(n, m)$

Then, $\hat{x}(k, l) = \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \hat{g}(p, q)\,\hat{w}(k-p, l-q)$ (Convolution of $g$ and $w$)

Note:

(Spatial domain) $I * g$ (Linear filtering: Linear combination of neighborhood pixel values)

$$\Big\downarrow \text{DFT}$$

(Frequency domain) $MN\,\hat{I} \underbrace{\odot}_{\text{pixel-wise multiplication}} \hat{g}$ (Modifying the Fourier coefficients by multiplication)

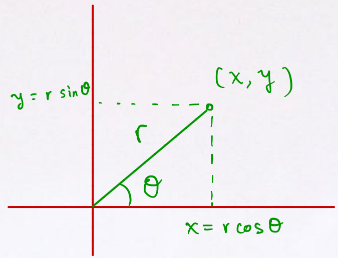
2. Average value of image

Average value of $g = \bar{g} = \dfrac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k,l) = \underbrace{\dfrac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi(0)}}_{\hat{g}(0,0)}$

3. DFT of a rotated image

Consider a $N \times N$ image $g$.

Then: $\hat{g}(m,n) = \dfrac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km+ln}{N}\right)}$




Write $k$ and $l$ in polar coordinates:

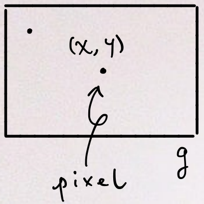
$k \equiv r\cos\theta \;;\; l = r\sin\theta$

Similarly, write $m \equiv \omega\cos\phi \;;\; n = \omega\sin\phi$.

Note that: $km + ln = r\omega(\cos\theta\cos\phi + \sin\theta\sin\phi) = r\omega\cos(\theta-\phi)$.

Denote $\mathcal{P}(g) = \{(r,\theta) : (r\cos\theta, r\sin\theta) \text{ is a pixel of } g\}$

(Polar coordinate set of $g$)




If $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, then $(r, \theta) \in \mathcal{P}(g)$.

Then: $$\underbrace{\hat{g}(m,n) = \hat{g}(\omega,\phi)}_{\text{Identify } \hat{g}(m,n) \text{ with } \hat{g}(\omega,\phi)} = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \underbrace{g(r,\theta)}_{\text{Identify } g(k,l) \text{ with } g(r,\theta)}\, e^{-j2\pi\left(\frac{r\omega\cos(\theta-\phi)}{N}\right)}$$

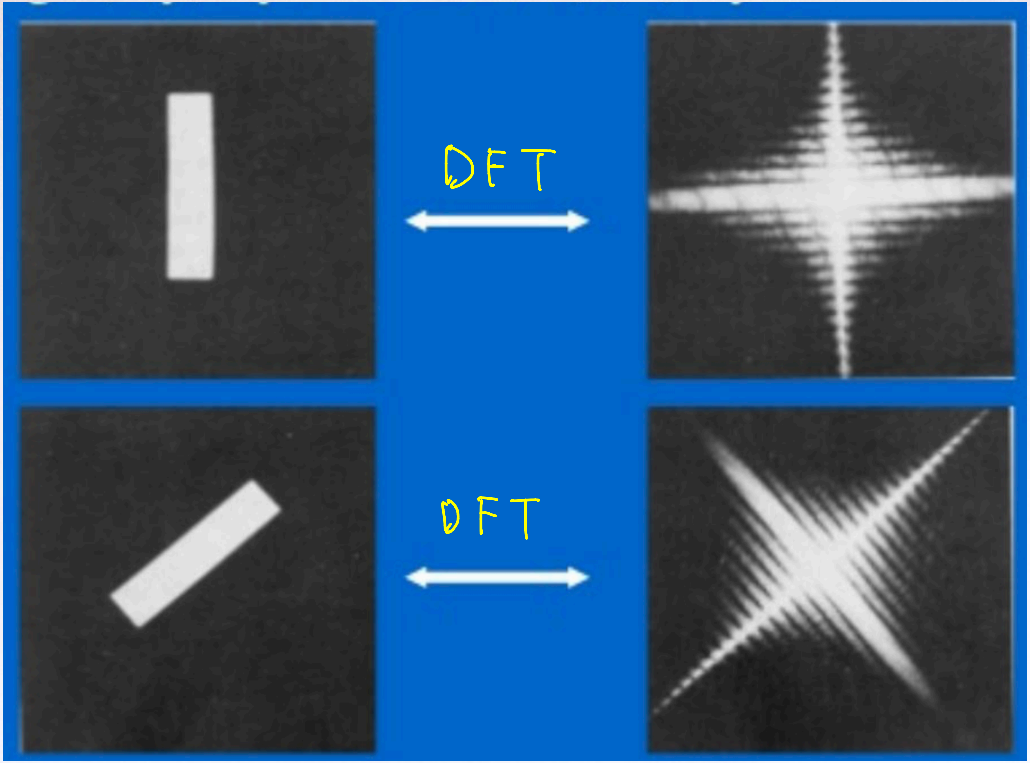
Consider a rotated image $\tilde{g}(r,\theta) = g(r, \theta+\theta_0)$ where $\theta$ is defined between $-\theta_0$ to $\pi/2 - \theta_0$.

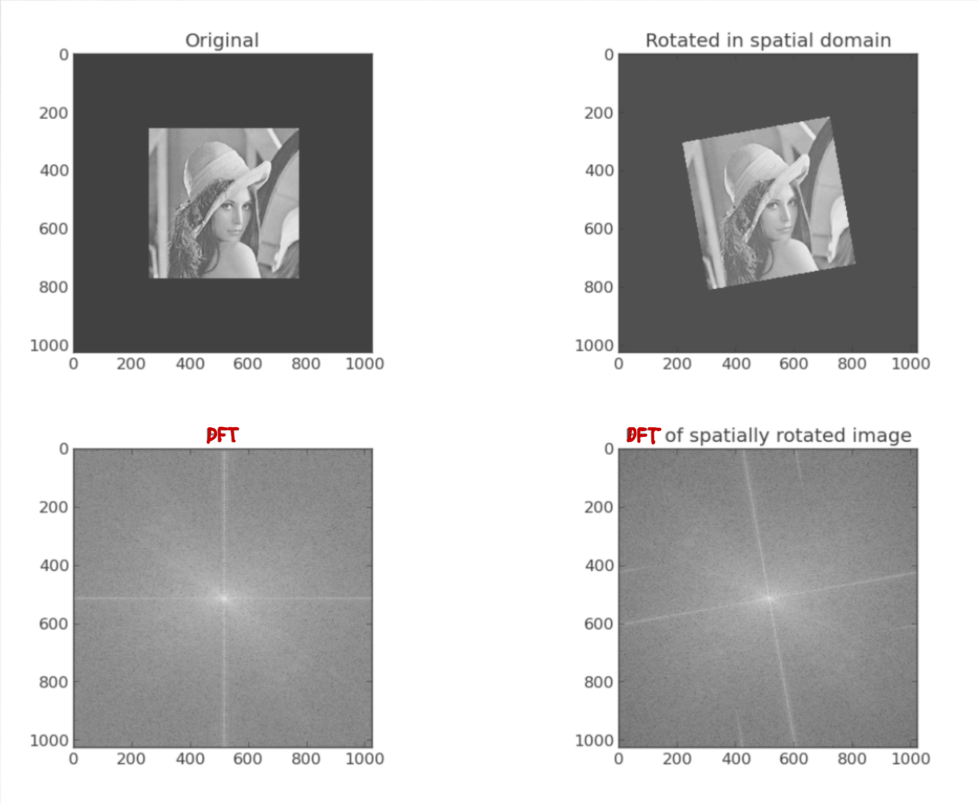
$\therefore$ image $g$ is rotated clockwisely by $\theta_0$.

DFT of $\tilde{g}$ is:

$$\hat{\tilde{g}}(\omega,\phi) = \frac{1}{N^2}\sum_{(r,\theta)\in\mathcal{P}(\tilde{g})} \underbrace{\tilde{g}(r,\theta)}_{= g(r,\, \underbrace{\theta+\theta_0}_{\tilde{\theta}})} e^{-j2\pi\left(\frac{r\omega\cos(\theta-\phi)}{N}\right)} = \frac{1}{N^2}\sum_{(r,\tilde{\theta})\in\mathcal{P}(g)} g(r,\tilde{\theta})\, e^{-j2\pi\left(\frac{r\omega\cos(\tilde{\theta}-\theta_0-\phi)}{N}\right)}$$

$\therefore \hat{\tilde{g}}(\omega,\phi) = \hat{g}(\omega, \phi+\theta_0)$. ( $\phi$ is also defined between $-\theta_0$ to $\pi/2 - \theta_0$)


DFT
DFT


Original
Rotated in spatial domain
DFT
DFT of spatially rotated image
0
200
400
600
800
1000